Topological Conformal Defects with Tensor Network for Tricritical Ising Model*

Tu Jingxin† and Yang Chen‡
Department of Physics, Fudan University
(Dated: June 9, 2019)

The tricritical Ising model is one of the few physically relevant theories endowed with supersymmetry.[1] In this article, we applied the tensor network to investigating the properties of tricritical Ising model in terms of symmetry and conformal field theory(CFT) to explore the topological defects of tricritical Ising model(TIM) using the methods tensor network renormalization(TNR). By comparing results obtained from TNR and the exact results, the efficiency of TNR while dealing with many-body system can be well demonstrated. Our work is to make exploration about the properties of tricritical Ising model and make adjustments to parameters and get numerical results.

I. INTRODUCTION

A conformal defect is a universality class of critical behavior at the junction of two critical systems. A topological conformal defect in a conformal field theory (CFT) is a particular type of conformal defect that is totally transmissive and can be deformed without affecting the value of correlators as long as it is not taken across a field insertion. It can also be regarded as defining a form of twisted boundary conditions for that CFT.[2]

Tensor Network Theory(TNT) is a kind of highly efficient and accurate method for simulating strongly correlated quantum systems. To evaluate partition functions and Euclidean path integral , Kadanoff's spin-blocking procedure opened the path to non-perturbative approaches based on coarse-graining a lattice.[3–6] In 2007, Levin and Nave proposed the tensor network group(TRG),[7] a versatile real-space coarse-graining transformations for 2D classical partition functions. TRG is an extremely useful approach that has revolutionized how we coarse-grain lattice models. However, TRG fails to remove part of the short-range correlations in the partition function, and as a result, leading to an RG flow with non-critical fixed points, which conflicts with the very spirits of the renormalization group(RG).[8] And in the third section we will introduce the techniques we adapted to the coarse-graining of partition functions/Euclidean path integrals, leading to a proper RG flow.[2] And after we get the results of fixed points, we can explore the properties of topological conformal defects.

The goal of this paper is to investigate the usage of tensor network techniques to describe topological conformal defects in tricritical Ising model. The tricritical Ising model turns out to have five non-trivial topological conformal defects. And we analyzed the symmetry defect $D_{\epsilon''}$ as an example.

A. Defects and transfer matrices

The statistical partition function Z of the tricritical Ising model on a square lattice with periodic boundary conditions, can be expressed as a $m \times n$ square which contain $m \times n$ sites. And at criticality, the TNR can be used to extract universal information about a phase transition by studying the partition function of a finite system. In a finite system one can observe a realization of the so-called operator-state correspondence in conformal field theories(CFT).[2] The operator-state correspondence says that all states in the theory can be created by operators which act locally in a small neighborhood of the origin. By the operator-state correspondence of CFT, this partition function can be expressed as a function of the scaling dimension Δ_α and conformal spins s_α of some specific set of scaling operator ϕ_α. And in the final results we draw the plot of conformal towers with three local primary fields of the "tricritical Ising CFT". We can extract $\{\Delta_\alpha, s_\alpha\}$ from the spectrum of eigenvalues of a transfer matrix M for the partition function Z, which fulfill the equation $Z = \mathrm{Tr}(M^m)$.

For a topological conformal defect D, by rewriting the partition function and transfer matrix as Z_D and M_D. Then we can extract another set of scaling dimension and conformal spins, making up of another conformal tower.

In this paper, the first step is to build tensor network representations of the transfer matrices and then we will analyze the process of extracting the scaling dimension and conformal spins $\{\Delta_\alpha\ , s_\alpha\}$. The final process is to diagonalizing the transfer matrices. And compared to diagonalizing the transfer matrices with exact diagonalization techniques, TNR has its superiority from aspect of dealing with problems in many-body systems, providing more accurate numerical results.

For concreteness, the process for Z without defects and Z_D exist some differences. For topological conformal defects, we manipulate a local unitary transformation that moves the location of the topological defect, producing a generalised translation operator T_D that commutes with M_D. The objects of TNR for topological defects are $T_D \cdot M_D$ of the defects.

* Thanks Professor Hung Ling-yan for her patience and thoughtful instruction. Thanks Professor Chen Yan for his encouragement and efforts, without him, we would never work this article out.
† jxtu17@fudan.edu.cn
‡ cyang16@fudan.edu.cn

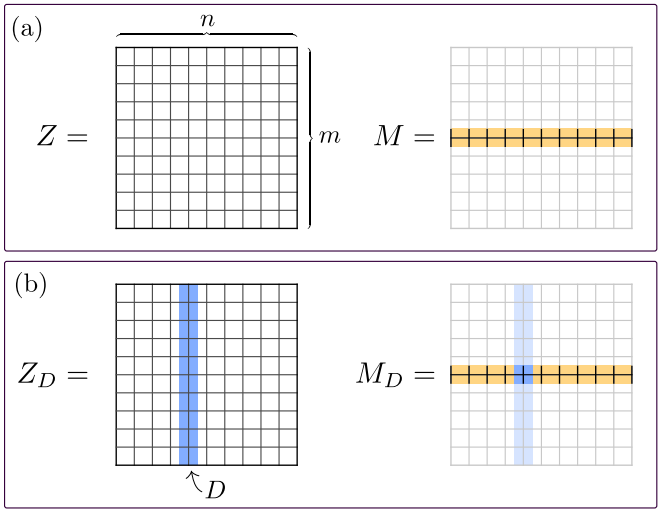

FIG. 1. (a)Partition function Z on lattice square made of $n \times m$ sites with periodic boundary conditions, and the corresponding transfer matrix M. (b)Partition function Z_D with the same periodic boundary conditions and the transfer matrix M_D. The blue column D is a defect boundary condition.[9]

B. Structure of this paper

Section II mostly describes the properties of tricritical Ising model. In details, section II includes the introduction of tricritical Ising model and describes some basic of minimal models in conformal field theory(CFT), which includes tricritical Ising CFT that effectively describes the lattice model at criticality. Then in section III, we mainly introduce the method of Tensor Network Renormalization(TNR) and we will focus on the specific process of applying this method to dealing with problems of statistical models. Section IV analyze a kind of topological superconformal defect $D_{\epsilon''}$. We express the partition functions $Z_{D_{\epsilon''}}$ of tricritical Ising model in terms of transfer matrices $M_{D_{\epsilon''}}$, and propose generalized translation operators $T_{D_{\epsilon''}}$. Rewriting all the above objects with tensor network representation, coarse-graining transformation for the products $T_{D_{\epsilon''}} \cdot M_{D_{\epsilon''}}$. By diagonalizing a transfer matrix that has been coarse-grained using TNR, we obtain a series of scaling dimension and conformal spins $\{\Delta_\alpha, s_\alpha\}$. Then we compare these numbers with the exact values to confirm the efficiency of the algorithm TNR, implementing in tricritical Ising model.

C. Source Code

The numerical results we present were obtained using a Python3 algorithm. The algorithm is mostly based on the source code at arXiv:1512.03846, licensed under the MIT License. We made some adjustment to obtain the numerical results of tricritical Ising model, while in the original codes they computed the numerical results of Ising model and Potts3 model. And to get some universal data of tricritical Ising model, we use Matlab and Mathematica to make some calculations. Thanks the authors of arXiv:1512.03846 for generously sharing the copyright to everyone with their source code and figures in their article.

II. TRICRITICAL ISING MODEL

The tricritical Ising model is the simplest example of superconformal field theory. Its Hilbert space contains a finite number of irreducible representations of the super-Virasoro algebra(the antiholomorphic part is omitted)[1]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}$$

$$\{G_m, G_n\} = 2L_{m+n} + \frac{1}{3}c(m^2 - \frac{1}{4})\delta_{m+n}$$

$$[L_m, G_n] = (\frac{1}{2}m - n)G_{m+n}$$

In particular, it is the simplest known statistical model to exhibit Supersymmetry, and the fact that it can be realized experimentally makes it an important model to study.[10]

In this section, we introduce the tricritical Ising model on lattice. We also introduce their continuum limit, and some other properties of tricritical Ising model as a minimal model in conformal field theory. The conformal data can be estimated using exact diagonalization. The accuracy of these estimates is limited by non-universal, finite-size correlations.

A. Partition Function

The tricritical Ising model is defined by its Hamiltonian

$$H[\sigma_i, t_i] = -\sum_{\langle i,j \rangle} t_i t_j (K + \delta_{\sigma_i,\sigma_j}) - \mu \sum_i t_i$$

where the variable $t_i = \sigma^2$ is 0 if site is vacant and 1 otherwise. K is the energy of a pair of unlike spins, and $K+1$ that of a pair of like spins. The chemical potential μ specifies the average number of occupied sites on the lattice. And σ_i, σ_j can take the values $\pm 1, 0$. There is another kind of expression where the Hamiltonian of tricritical Ising model is

$$H[\sigma_i, \sigma_j] = -\sum_{\langle i,j \rangle} J\sigma_i\sigma_j - \mu \sum_i (\sigma_i)^2$$

. Here $\sigma_i, \sigma_j = \pm 1, 0$.

The partition function at inverse temperature β is

$$Z = \sum_{ij} e^{\beta H} = \sum_{ij} e^{-\beta(\sum_{\langle i,j\rangle} t_i t_j (K+\delta_{\sigma_i,\sigma_j}) - \mu \sum_i t_i)}$$

The chemical potential is a function of temperature, which means there are relationship of $e^{\mu} = f(T)$.[10] According to the theory of RG flow, whatever value we set for μ, with TNR methods it will efficiently flow to the fixed point. So that in codes we set $\mu = 0$. At some value of (β, K, μ), there is a critical point at which three phases meet and coexist critically. In addition to identity operator, five other scaling operators emerge at this tricritical point: three energy-like operators $\sigma, \sigma', \sigma''$, corresponding to the three terms of the configuration energy, and two spin-like operators ϵ, ϵ'.[1]

The class of universality tricritical points occurring in two-dimensional statistical models whose order parameter enjoys $\mathbb{Z}_2$ symmetry, is described (in the scaling limit) by the minimal conformal field theory characterized by central charge $c = \frac{7}{10}$.[1] For low vacancies and low temperature, we have an ordered phase I with spontaneously broken $\mathbb{Z}_2$ symmetry. On the other side of the transition line we have a disordered phase II with unbroken symmetry. At β_c there is a phase transition that separates the low temperature symmetry-breaking, orderd phase from the high temperature disordered phase. And we set a approximation value of β_c during a specific computation.

B. Tricritical Ising CFT

As we have mentioned above, the classical square lattice model has a critical point at β_c. Behaviors near critical points and continuum limits of these critical points are both described by *minimal models*, the particularly simple conformal theories. Considering the twisted boundary conditions, on account of the same class of symmetry, methods applied on 2-dimensional Ising model can be applied on the tricritical Ising model. Therefore, the universal properties are obtained from the conformal data. Specifically, according to the periodic boundary conditions defined by a complex modular parameter $\tau = \tau_1 + i\tau_2$, the partition function of a CFT is

$$Z_{CFT} = \mathrm{Tr}(e^{-2\pi\tau_2(L_0+\bar{L_0}-\frac{c}{12})} e^{2\pi i\tau_1(L_0-\bar{L_0})})$$

$$= \mathrm{Tr}(e^{-2\pi\tau_2 H_{CFT}} e^{2\pi i\tau_1 P})$$

. Here L_0 and $\bar{L_0}$ are the Virasoro generators and H_{CFT} and P are the Hamiltonian and momentum operators. c is the central charge.

The scaling operators ϕ_α of the CFT are eigenoperators of dilations on an infinite plane. The operator-state correspondence identifies them with states $|\phi_\alpha\rangle$ that are the eigenvalues of L_0 and $\bar{L_0}$: $L_0|\phi_\alpha\rangle = h_\alpha|\phi_\alpha\rangle$ and $\bar{L_0}|\phi_\alpha\rangle = \bar{h}_\alpha|\phi_\alpha\rangle$. h_α and $\bar{h}_\alpha$ are known as the holomorphic and antiholomorphic conformal dimension of ϕ_α.[9]

Then the partition function can be rewritten as

$$Z_{CFT} = \sum_\alpha e^{-2\pi\tau_2(h_\alpha+\bar{h}_\alpha-\frac{c}{12})+2\pi i\tau_1(h_\alpha-\bar{h}_\alpha)}$$

$$= \sum_\alpha e^{-2\pi\tau_2(\Delta_\alpha-\frac{c}{12})+2\pi i\tau_1 s_\alpha}$$

where $\Delta_\alpha = h_\alpha + \bar{h}_\alpha$ and $s_\alpha = h_\alpha - \bar{h}_\alpha$ are known as the scaling dimension and conformal spins of ϕ_α. The conformal towers are built up from the primary operators such as h_α and $\bar{h}_\alpha$, and then conformal towers are built up with scaling dimension of the form $h = h_\alpha + k$ and $\bar{h} = \bar{h}_\alpha + l$, where $k, l \in \mathbb{N}$. The theoretical derivation is the specific realization of operator-state correspondence theory.

The tricritical Ising model CFT is a conformal field theory of central charge $c = \frac{7}{10}$. The tricritical Ising model includes six "diagonal" primary operators that have $h = \bar{h}$. They are called the identity $\mathbb{1}$ for (0,0), the magnetization σ for $(\frac{3}{80}, \frac{3}{80})$, the energy density ϵ for $(\frac{1}{10}, \frac{1}{10})$, the submagnetization σ' for $(\frac{7}{16}, \frac{7}{16})$, the chemical potential ϵ' for $(\frac{3}{5}, \frac{3}{5})$, a irrelevant primary operator ϵ'' for $(\frac{3}{2}, \frac{3}{2})$. Because of the $\mathbb{Z}_2$ symmetry of the model, the conformal towers come with a parity. This parity is +1 for $\mathbb{1}, \epsilon, \epsilon''$ and ϵ', and -1 for σ and σ'.[11]

In section IV, we will see that the non-diagonal combinations of h and $\bar{h}$ are relevant to the topological conformal defects.

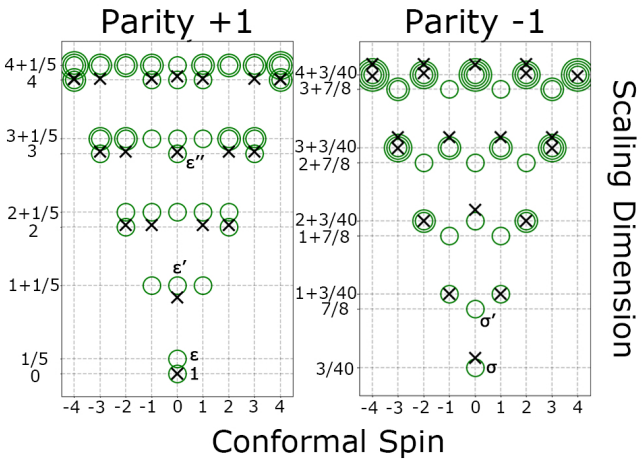

FIG. 2. Scaling dimension and conformal spins of tricritical Ising model without defects. We run the code with setting bond dimension χ, χ' as 11,22 and coarse-graining 9 times in total. Compared with the results using less coarse-graining steps, the numerical results of scaling dimension are more accurate. With additional coarse-graining the accuracy of conformal spins increases to a high level. To get a better graphic result, we can put the accurate numbers into one graph. In this figure we can find primary identity $\mathbb{1}(0,0)$, $\epsilon'(\frac{3}{5}, \frac{3}{5})$, $\epsilon''(\frac{3}{2}, \frac{3}{2})$, $\sigma(\frac{3}{80}, \frac{3}{80})$, and vacuum for σ', ϵ.

III. NUMERICAL RESULTS

The numerical results we obtained from computation of tricritical Ising model without defects. And the results are shown in Table I.

Primary	$(h, \bar{h})$	Δ_{TNR}	Δ_{exact}	s_{TNR}	s_{exact}
$\mathbb{1}$	$(0,0)$	0	0	10^{-16}	0
ϵ	$(\frac{1}{10}, \frac{1}{10})$	*vacant*	0.2	*vacant*	0
ϵ'	$(\frac{3}{5}, \frac{3}{5})$	1.0367537	1.2	-10^{-17}	0
ϵ''	$(\frac{3}{2}, \frac{3}{2})$	3.0143559	3	0	0
σ	$(\frac{3}{80}, \frac{3}{80})$	0.2056824	0.075	-10^{-16}	0
σ'	$(\frac{7}{16}, \frac{7}{16})$	*vacant*	0.875	*vacant*	0

TABLE I. The scaling dimension and conformal spins of the primary fields of the tricritical Ising model obtained using TNR, contrasted with the exact values.

IV. TENSOR NETWORKS

First, we introduce how to use tensor networks to express the partition function Z and its transfer matrix M. Then we will describe how to use a coarse-graining algorithm for tensor networks to analyze large systems followed with exact diagonalization. By diagonalizing a transfer matrix M in respect of a large number of n of sites we can significantly reduce the errors due to finite-size corrections in estimating scaling dimension and conformal spins $\{\Delta_\alpha, s_\alpha\}$.[9] In detail, we will introduce the superiority of the TNR algorithm in terms of reducing the truncation errors and speeding up the procedure of get critical fixed points.

A. Tensor Network Representation

For tricritical Ising model, with its partition function we can write out the Boltzmann weights

$$B_{ij} = e^{\beta t_i t_j (K + \delta_{\sigma_i \sigma_j})}$$

where we don't take consideration of chemical potential μ in this situation. Then the partition function is

$$Z = \sum_{\{\sigma\}} \prod_{\langle i,j \rangle} B_{ij}$$

, summing over all the product of all the spin configurations and nearest-neighbor pairs. Periodic boundary conditions in both directions are assumed. This partition function Z can be written as a tensor network in many ways. Here we use the notation presented in *Topological conformal defect with tensor network* in Figure.3. The first Figure.3(a) is a direct translation from the equation above into a tensor network graph notation. For each spin there is a four-index Kronecker delta δ_{ijkl}. The matrix B connected two neighbor leg of two spins. Then we can see it from the graph that we can use the notation $A_{ijkl} = B_{ij} B_{jk} B_{kl} B_{li}$ to represent the interactions around the spin. A_{ijkl} can be regarded as the initial tensor. Every index of A presents a spin while each tensor accounting for two spins. The original square lattice has been rewritten to a tensor network $Z_{m,n}(A)$.The rows and columns of A are denoted as m and n.

It needs to be clarified that when we read equations or tensor network diagrams, we read the equations from left to right and read the diagrams from left to right or bottom to top.

We can see it from Figure.3 that Z can be rewrite as $Z = Tr(M^m)$ where the transfer matrix is in Figure.3(b), or

$$M^{k_1 k_2 \ldots k_n}_{j_1 j_2 \cdots j_n} = \sum_{i_1, i_2 \ldots i_n} \prod_{\alpha=1}^{n} A_{i_\alpha j_\alpha i_{\alpha+1} k_\alpha}$$

Here all the i_α are summed over and i_1 is identical with i_{n+1}. We here assumed M is a linear map from $\mathbb{V}_j$ to $\mathbb{V}_k$(noted that $\mathbb{V}_j$ is the tensor product of all the vector spaces of indices j_α(as well as $\mathbb{V}_k$ and k_α))[9].

Figure.3(c) is presenting a lattice translation in the network. The operator T is used to connect the boundary of the rewritten square lattice. In Figure.3(d) is the operator $T \cdot M$, which we need to diagonalize to extract universal data of phase transition.

The $\mathbb{Z}_2$ symmetry of tricritical Ising model is crucial in the tensor network representation. For a model with global internal symmetry, the symmetry can be made manifest in the tensors themselves.[7] For now we only need to know that for tricritical Ising model this means that we can use the methods shown in Figure.4, which means if we add a spin-flip matrix V on each leg, the tensors are left unchanged. It is a unitary matrix that contains $V^2 = \mathbb{1}$. We call a tensor as $\mathbb{Z}_2$ invariant tensor if it obeys the invariance property shown in Figure.4. For tricritical Ising model, we set

$$V = \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} & \sqrt{2} \\ -\frac{\sqrt{6}}{3} & \frac{2\sqrt{3}}{3} & 1 \\ \sqrt{2} & 1 & 0 \end{pmatrix}$$

.The vector space connected to each leg of a $\mathbb{Z}_2$ invariant tensor is the direct sum of two subspaces, one for each parity ± 1. Thus we can add a parity to each vector of the transfer matrix and the corresponding scaling operator ϕ_α.

B. Coarse-graining

A tensor network coarse-graining transformation maps a network like $Z_{n,m}(A)$ to a network $Z_{n',m'}(A')$, with its elements A' and describes a longer length scale features

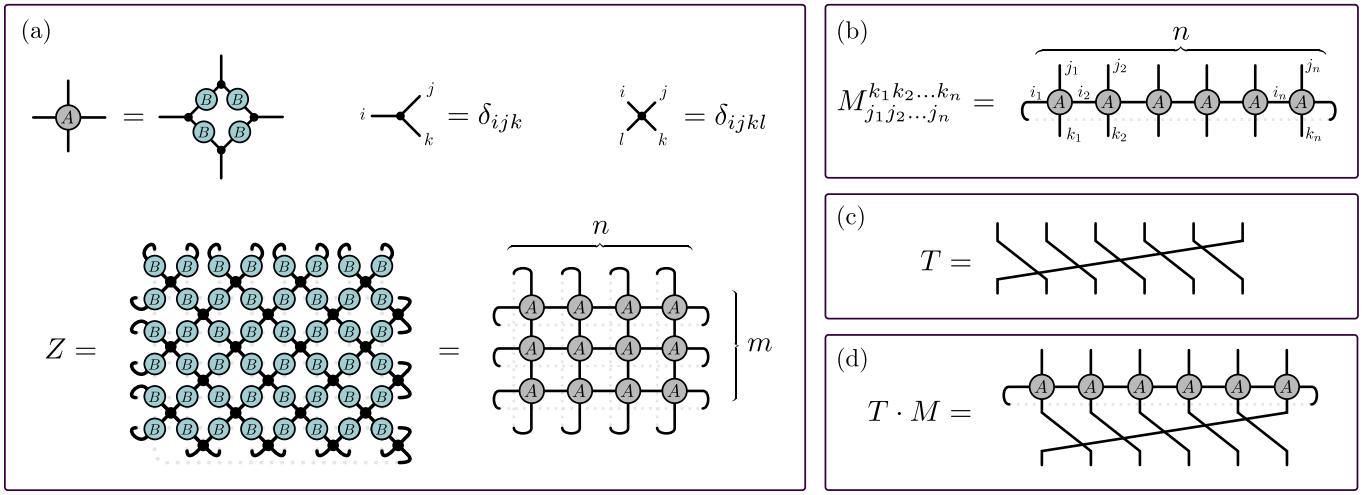

FIG. 3. Graphic tensor network language.(a)The graphic presentation of a single tensor A and the translation of partition function from a square lattice to a tensor network.(b)Translation of transfer matrix in tensor network representation.(c)The one-site translation operator T (d)The translation operator composed with the transfer matrix[9]

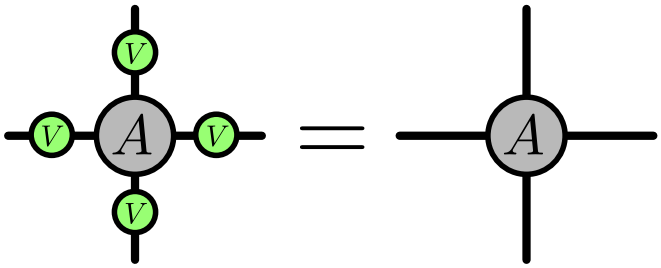

FIG. 4. The invariance of tensor A under symmetry transformation V[9]

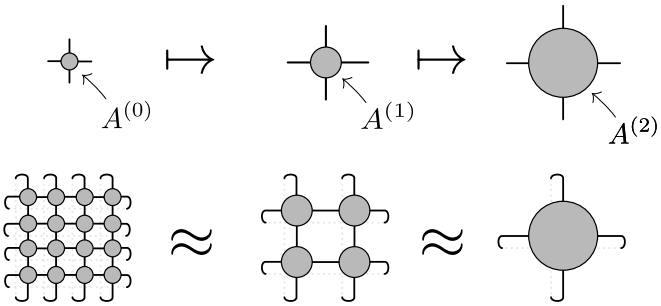

FIG. 5. A coarse-graining procedure which present RG flow.[9]

of the system. Here for instance we consider a coarse-graining where each A' stands for four original tensors A and $n' = \frac{n}{2}$, $m' = \frac{m}{2}$, which means the linear size of the system downsized by $\frac{1}{2}$.

Using the same procedure of coarse-graining, we obtain $A^{(0)} \mapsto A^{(1)} \mapsto \ldots \mapsto A^{(s)}$ that $A^{(0)} \equiv A$. After several times of repeated coarse-graining the tensor converged to a fixed-point tensor $A^{(s)}$. For each tensor $A^{(s)}$, it represents 4^s of the original tensors $A^{(0)}$, and the network $Z_{\frac{n}{2^s}, \frac{m}{2^s}}(A^{(s)})$ is an approximation to the original network, which can be seen in the Figure.5. Then this tensor network can be used to describe the properties of tricritical Ising model near criticality according to the theory of renormalization group. We can then use the coarse-graining tensors $A^{(s)}$ to produce a transfer matrix M representing many spins and extract $\{\Delta_\alpha, s_\alpha\}$ with smaller finite-size corrections.[9] That's the basic procedure of tensor network coarse-graining. However, when it comes to the specific steps of coarse-graining, each step brings some truncation errors. To reduce the truncation errors, G.Evenbly and G.Vidal have developed an efficient method-tensor network renormalization(TNR). In details, the method is based on inserting approximate partitions of unity into the network and optimizing them to minimize the truncation error. By this way, in each step it removes all short- range correlations and realize a proper renormalization group flow with the critical fixed point tensor. The detailed procedure has been presented in Figure.6.

There is an essential role in the coarse-graining process called the bond dimension of the network. It's the dimension χ of the indices of $A^{(s)}$. The bond dimension controls the computational cost of coarse-graining, which grows as a power of χ, as well as the truncation errors introduced at each coarse-graining step decreasing by growing χ. In order to estimate $\{\Delta_\alpha, s_\alpha\}$, a useful coarse-graining scheme is then one where a sufficiently small χ can be kept over several coarse-graining steps while at the same time keeping the truncation errors sufficiently small, so that they do not significantly affect the numerical estimates.[2] We can find out that the best parameters are the parameters which the finite-size corrections and truncation errors are of the same magnitude, and their cumulative effect on the results is thus at the minimum.

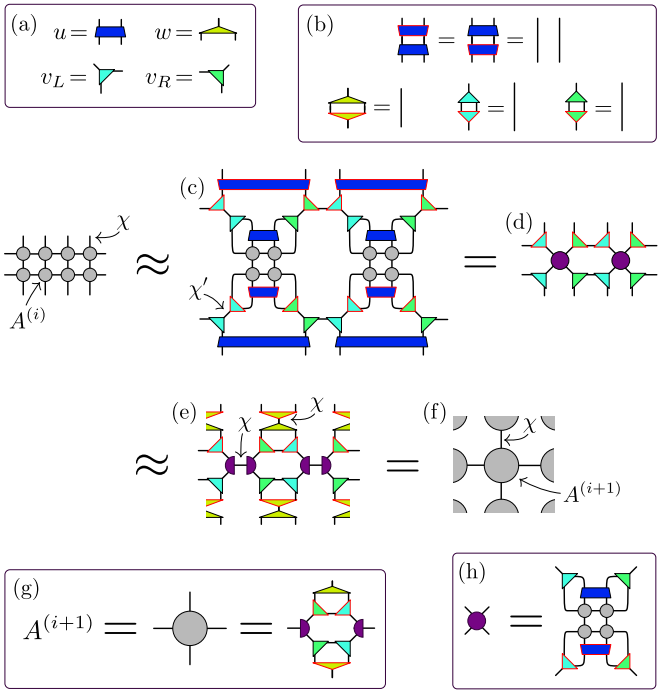

FIG. 6. The picture shows the detailed procedure of a step of coarse-graining in TNR (a)Show the tensors used to optimizing coarse-graining procedure in TNR.(b)the u tensor is unitary and the w, v_L, v_R tensors are isometric.(c)Inserting u, v_L, v_R with truncation errors, by optimizing the auxiliary tensors to reduce the errors.(d)(h)Contracted tensors.(e)(f)Inserting w and the purple tensors are split into two parts. Bond dimension $\chi, \chi^{'}$ are the (maximum allowed) dimension of the bonds they point at and all similar bonds in the network.[9]

Applying a $2 \times 2 \mapsto 1$ coarse-graining transformation s times, then from a $2^s \times (2^s \cdot n_s)$ tensors A we obtain a transfer matrix $M_{(s)}$ in Figure.6. The computational cost then scales logarithmically in system size, which means $M^{(s)}$ with dimension $\chi^{(n_s)} \times \chi'^{(n_s)}$ can be diagonalized for sufficiently small values of n_s and χ. In the same way we diagonalize $T^{(s)} \cdot M^{(s)}$ where $T^{(s)}$ is a translation operator. After the final coarse-graining step on the composite operator $T^{(s)} \cdot M^{(s)}$, the periodicity of the conformal spins is raised to $2n_s$.[9]

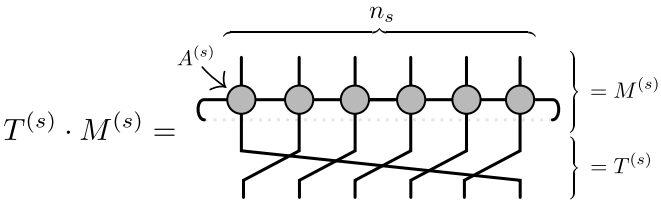

FIG. 7. The coarse-grained transfer matrix and translation operator[9]

V. TOPOLOGICAL DEFECTS

Different topological conformal defects can be thought of as different boundary conditions, leading to different partition functions. The tricritical Ising model has five different topological defects, while we will introduce them in the following part generally. And in next section, we mostly introduce one type of topological defect which we have realized the conformal data with TNR method by codes.

As we have introduced earlier in section II, the partition function of a CFT on a torus can be written as:

$$Z_{CFT} = \mathrm{Tr}(e^{-2\pi\tau_2 H_{CFT}} e^{2\pi i \tau_1 P})$$

Considering a twisted partition function Z_D as:

$$Z_D = \mathrm{Tr}(D e^{-2\pi\tau_2 H_{CFT}} e^{2\pi i \tau_1 P})$$

Here D is a twist operator, which represents a special type of boundary condition. If D commutes with all the generators of the Virasoro algebra it is called a topological conformal defect.[9] If the defect is topological the loop can be freely deformed without affecting correlation functions in the systems as long as the defect is not moved across a field insertion. The conformality of the defect also means that it is invariant under scale transformations, namely in the process of TNR, the defect is invariant.

The twisted partition function Z_D can be written as a sum of terms relating to scaling dimension and conformal spins $\{\Delta_\alpha, s_\alpha\}_D$, with the same manipulation as non-twisted Z.[9]

The two defects next to each other can behave as one defect, which obey the fusion rules of the topological defects as follows,

$$D_{\epsilon''} \times (D_{\mathbb{1}}, D_\epsilon, D_{\epsilon'}, D_{\epsilon''}, D_\sigma, D_{\sigma'}) = (D_{\epsilon''}, D_{\epsilon'}, D_\epsilon, D_{\mathbb{1}}, D_\sigma, D_{\sigma'})$$

Like Ising CFT, all possible topological conformal defects of tricritical Ising model can be written as linear combination of six defects which we have mentioned. And these simple defects can be expressed as $D_{\mathbb{1}}, D_\epsilon, D_{\epsilon'}, D\epsilon'', D_\sigma, D_{\sigma'}$. They are related to the same irreducible representations of the Virasoro algebra as the primary field $\mathbb{1}, \epsilon, \epsilon', \epsilon'', \sigma, \sigma'$. The $D_{\mathbb{1}}$ defect is the trivial defect where the twist operator is just the identity. The partition function $Z_{D_{\mathbb{1}}} = Z$, has been discussed in previous sections. First of all, there is the $\mathbb{Z}_2$ symmetry related to the spin-reversal transformation, that in the Landau–Ginzburg approach corresponds to $\phi \to -\phi$. The fields $, \epsilon, \epsilon', \epsilon''$ are even with respect to such a transformation, while σ, σ' are odd. Another symmetry multilated from the lattice model is the Kramers–Wannier duality, under which the magnetization operators σ, σ' are mapped onto their corresponding disorder operators μ, μ', while ϵ, ϵ'' are odd, and ϵ' is even. The behavior of primary operators of TIM under these two discrete symmetries is summarized in Table II.[11]

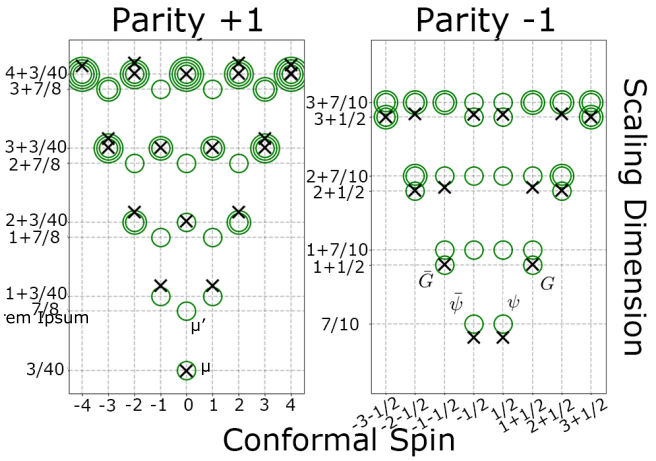

FIG. 8. Scaling dimension and conformal spins of tricritical Ising model with $D_{\epsilon''}$ defects. We run the code with setting bond dimension χ, χ' as 11,22 and coarse-graining 9 times in total.

Field	Spin-reversal	Kramers-Wannier
ϵ	ϵ	$-\epsilon$
ϵ'	ϵ'	ϵ'
ϵ''	ϵ''	$-\epsilon''$
σ	$-\sigma$	μ
σ'	$-\sigma'$	μ'

TABLE II. Discrete symmetries of TIM

The operators present in the twisted partition functions come organized in conformal towers built on top of primary operators, each of which is identified with conformal dimension$(h, \bar{h})$.[9] According to the fusion rules of primary operators of $M(5,4)$, we can work out the fusion rules of topological defects.

Field	Physical Meaning	Conformal Dimension
μ	Disorder Field	$(\frac{3}{80}, \frac{3}{80})$
μ'	Subleading Disorder Field	$(\frac{7}{16}, \frac{7}{16})$
ψ	Fermion	$(\frac{3}{5}, \frac{1}{10})$
$\bar{\psi}$	Anti-fermion	$(\frac{1}{10}, \frac{3}{5})$
G	Susy Generator	$(\frac{3}{2}, 0)$
$\bar{G}$	Susy Generator	$(0, \frac{3}{2})$

TABLE III. Operators in the TIM with $\mathbb{Z}_2$-twisted boundary conditions[11]

VI. SYMMETRY DEFECT: $D_{\epsilon''}$

To understand the $D_{\epsilon''}$ defect, through universal data we can extract the exact scaling dimension and conformal spins, which is shown in the graphic results with green circles. The topological defect comes from the twisted boundary condition, The conformal boundary conditions for $c = \frac{7}{10}$ may be classified following the work in [12]. According to the computation results in [10], The boundary states corresponding to "physical" boundary conditions can be constructed following the procedure in [12] and have the form

$$|\frac{\tilde{3}}{2}\rangle = C|0\rangle + \eta|\frac{1}{10}\rangle + \eta|\frac{3}{5}\rangle + |\frac{3}{2}\rangle - \sqrt[4]{2}|\frac{7}{16}\rangle - \sqrt[4]{2}|\frac{3}{80}\rangle$$

where $C = \sqrt{\frac{\sin\frac{\pi}{5}}{\sqrt{5}}}$ and $\eta = \sqrt{\frac{\sin\frac{2\pi}{5}}{\sin\frac{\pi}{5}}}$. Here $|j\rangle = \sum_N |j,N\rangle \otimes |j,N\rangle$ where j labels a highest weight representation of the algebra of the L_N, and $|j,N\rangle$ is an orthonormal basis in this representation space.[10]

But here to construct the Hamiltonian and partition function for $D_{\epsilon''}$ defect, we have to describe the boundary condition from the aspect of quantum spin chain. We have to introduce the spin-1 Blume-Capel Model(BCM) Quantum Spin Chain first. This model is obtained by the time-continuum limit of the well known BCM in two dimension. It describes the dynamics of spin-1 localized particles, with Hamiltonian given by

$$H_{BC} = -\sum_j (s_j^z s_{j+1}^z - \delta(s_j^z)^2 - \gamma s_j^x)$$

where s^s and s^z are the spin-1 $SU(2)$ operators. For values $\gamma > \gamma_{tr}$ the Hamiltonian has a quantum critical line $\delta_c(\gamma)$ governed by a CFT in the same universality class of the quantum Ising chain, i.e. Central charge $c = \frac{1}{2}$. At γ_{tr} the model has a quantum tricritical point at δ_{tr} in the universality class of the tricritical Ising model, having central charge $c = \frac{7}{10}$. For $\gamma < \gamma_{tr}$ there is a line $\delta = \delta_{gap}(\gamma)$ of first-order phase transitions. By accurately estimating, the tricritical point was located at $\gamma_{tr} = 0.41563$ and $\delta_{tr} = 0.91024$.[13] By setting the parameters of defect $D_{\epsilon''}$ in the codes and make comparison of the graphic results with the primaries of $D_{\epsilon''}$, we deduce that the boundary condition in the form of quantum chain can be described as

$$H_{D_{\epsilon''}} = -\sum_{j=1}^{n-1} (s_j^z s_{j+1}^z - s_n^z s_1^z - \delta_{tr}(s_j^z)^2 - \gamma_{tr} s_j^x)$$

. That's the lattice representation of the defect. If we manipulate the $H_{D_{\epsilon''}}$ with unitary operator, we can move the defect to one side.

As for defect $D_{\epsilon''}$, the $H_{D_{\epsilon''}}$ and $Z_{D_{\epsilon''}}$ can be denoted with tensor network representation like the Figure.9. The figure shows how the defect be moved to a side by conjugating some unitary operators.

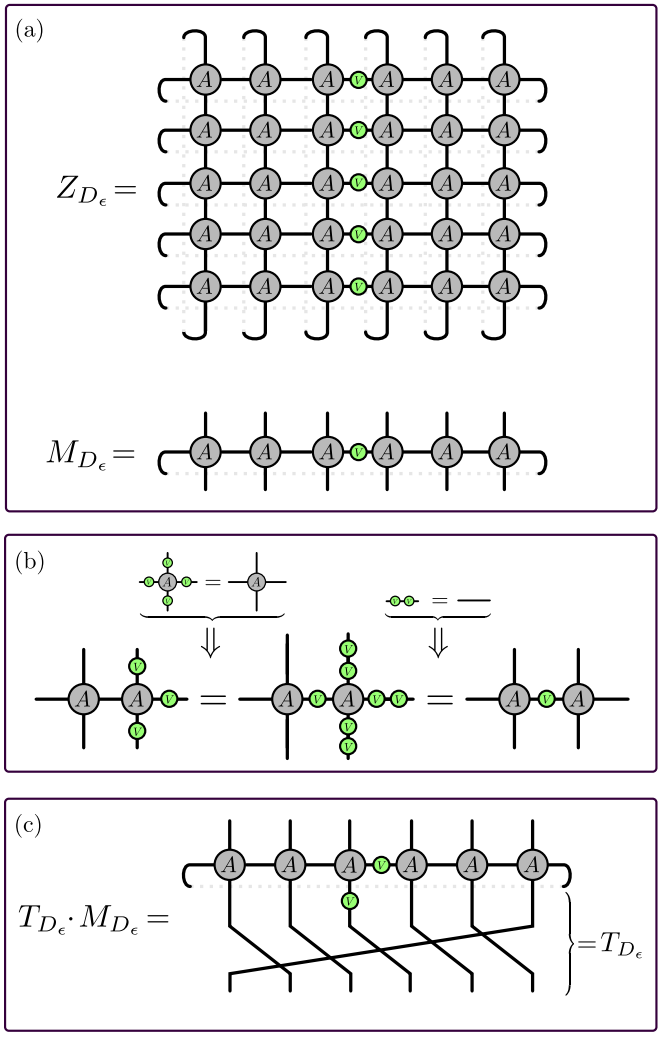

FIG. 9. (a)The tensor networks for the partition function $Z_{D_{\epsilon''}}$ and its transfer matrix $M_{D_{\epsilon''}}$ which fulfills $Z_{D_{\epsilon''}} = \mathrm{Tr}(M_{D_{\epsilon''}})$. V is the spin-1 $SU(2)$ operator s^x that flips a spin. (b)The invariance property of A and the fact that $V^2 = 1$ indicates that the symmetry defect can be removed to one side in this way. (c)The operator $T_{D_{\epsilon''}} \cdot M_{D_{\epsilon''}}$ whose diagonalization produces the conformal data $\Delta_\alpha, s_{\alpha_{D_{\epsilon''}}}$.[9]

By making some adjustment to the codes, we obtain a graphic result using the simulation with algorithm TNR as Figure.8. We can find that some primary operators emerge in the figure, they are the fermion field $\psi(\frac{3}{5}, \frac{7}{10})$, , antifermion field $\bar{\psi}, (\frac{7}{10}, \frac{3}{5})$, supersymmetry generators $G(\frac{3}{2}, 0)$ and $\bar{G}(0, \frac{3}{2})$ and the disordered field $\mu(\frac{3}{80}, \frac{3}{80})$. But the subleading disordered field has not appeared.

VII. NUMERICAL RESULTS

The numerical results are shown in Table V.

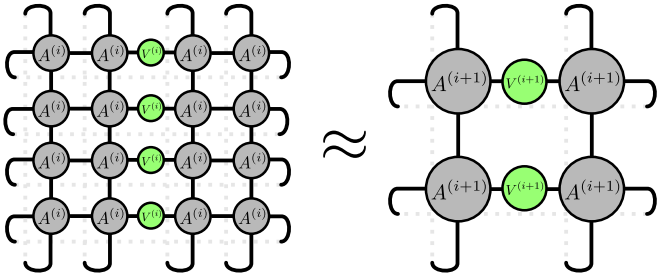

FIG. 10. Coarse-graining a $D_{\epsilon''}$ defect produces a similar defect at next scale. $A^{(i)}$ and $A^{(i+1)}$ are the same tensors when coarse-graining a system without a defect. [9]

$\mathbb{Z}_{D_1}$	0	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{2}$	$\frac{3}{80}$	$\frac{7}{16}$
0	1					
$\frac{1}{10}$		ϵ				
$\frac{3}{5}$			ϵ'			
$\frac{3}{2}$				ϵ''		
$\frac{3}{80}$					σ	
$\frac{7}{16}$						σ'

$\mathbb{Z}_{D_{\epsilon''}}$	0	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{2}$	$\frac{3}{80}$	$\frac{7}{16}$
0				G		
$\frac{1}{10}$			ψ			
$\frac{3}{5}$		$\bar{\psi}$				
$\frac{3}{2}$	$\bar{G}$					
$\frac{3}{80}$					μ	
$\frac{7}{16}$						μ'

TABLE IV. The primary operators included in tricritical Ising model partition functions with defect $D_{\epsilon''}$.

VIII. DISCUSSION

In this paper, we introduce the algorithm tensor network renormalization and tricritical Ising model. The theoretical analysis are mostly based on conformal field theory(CFT). And based on the previous work of 2-dimensional classical Ising model, our main work is to learn the relevant tricritical Ising CFT, specifically theories of *minimal model* and learn more about tensor network renormalization. Then we get two figures of conformal towers, which describing the differential value between the conformal data extracted from the classical computation and the data worked out by TNR, and some numerical results obtained from the codes. We have spent plenty of time to know more about the relevant theory under condition of lacking many background knowledge. And we spent much time on figuring out the algorithm of the source codes. On the basis of learning, we make adjustment for the codes and get some results that is under other expectation basically.

When it comes to the numerical results, we think that the main differences between the exact diagonalization results and the simulation results are from the imperfection in some aspects of the algorithm, so that the graphic results lack some primary operators. As for the original parameters, with the bond dimension χ, χ' increasing, the computation times are increasing exponentially and truncation errors are decreasing quickly, while with the coarse-graining times increasing, the truncation errors are increasing and the accuracy of the accuracy of

Primary	$(h, \bar{h})$	Δ_{TNR}	Δ_{exact}	s_{TNR}	s_{exact}
μ	$(\frac{3}{80}, \frac{3}{80})$	0.0694909	0.075	10^{-16}	0
μ'	$(\frac{7}{16}, \frac{7}{16})$	*vacant*	0.875	*vacant*	0
ψ	$(\frac{1}{10}, \frac{3}{5})$	0.5183355	0.7	0.4999737	0.5
$\bar{\psi}$	$(\frac{3}{5}, \frac{1}{10})$	0.5183355	0.7	-0.4999737	-0.5
G	$(0, \frac{3}{2})$	1.5064868	1.5	1.4999563	1.5
$\bar{G}$	$(\frac{3}{2}, 0)$	1.5064868	1.5	-1.4999563	-1.5

TABLE V. The scaling dimension Δ and conformal spins s for the primaries of $Z_{D_{\epsilon''}}$ as obtained with TNR compared with the exact values. Here we present the numerical results to demonstrate the accuracy of our method.

conformal spins increases to a high level while with less coarse-graining times, we will get more accurate simulation of scaling dimension. Because of time limitation, we set the bond dimension χ, χ' as 11,22 and to get each graphic result we compute for nearly one day and in this paper we present the results which still seems not so fixed with exact results.

In the future, we have the following aspects to dig more, first we plan to explore the relevant conformal field theory(CFT) in-depth knowledges and implement the method on other topological conformal defects or other minimal models. For concreteness, we plan to dig more about the boundary condition of the other defects and optimize our algorithm. Besides, we plan to explore the tensor network renormalization methods and apply it to other aspects by programming on the basis of deep exploration of the source codes at arXiv:1512.03846.

[1] P. F. P. D.Senechal, *Conformal Field Theory* (Springer, 1997).
[2] G. G.Evenbly, "Tensor network renormalization," (2015), arXiv: 1412.0732v3.
[3] K. Wilson, Adv.Math **16** (1975).
[4] K. Wilson, Rev. Mod. Phys **47** (1975).
[5] S. White, Phys. Rev. L **69** (1992).
[6] S. White, Phys. Rev. B **48** (1993).
[7] M. Levin, Phys. Rev. L **99** (2007).
[8] X. W. Z.C. Gu, Phys. Rev. B **80** (2009).
[9] e. a. M.Hauru, G.Evenbly, "Topological conformal defects with tensor networks," (2016), arXiv: 1512.03846v3.
[10] L. Chim, "Boundary s-matrix for the tricritical ising model," (1995), arXiv:hep-th/9510008.
[11] G. M. Alessandro Mossa, "Analytic properties of the free energy: the tricritical ising model," (2008), arXiv: 0710.0991v2.
[12] J.L.Cardy, Nuclear Physics B **324** (1989).
[13] F. C. A. J. C. Xavier, "Finite-size corrections of the entanglement entropy of critical quantum chains," (2011), arXiv:1111.6577.

IX. APPENDIX

First of all, we are deeply grateful for the guidance and suggestions during the process from professor Hung Ling-yan. We are grateful for her proposing this problem while we ask for some advice when we have no ideas for the task. It's a great opportunity for us to begin learning about tensor network and we hope to continue digging on it. Thanks Professor Chen Yan for his encouragement for our work and proving the opportunity for us to start such a task. The process is demanding but meaningful very well.

Here are some key codes of our source code. It's a pity that we cannot put the entire code here because of its length(about 2×10^4 lines!)

```python
def tricritical_ising_hamiltonian(pars):
    ham = (- pars["J"]*np.array([[1, 1, 0],
                                 [1, 1, 0],
                                 [0, 0, 0]],
                                dtype=pars["dtype"])
           - np.array([[1, 0, 0],
                       [0, 1, 0],
                       [0, 0, 0]],)
           + pars["H"]*np.array([[1, 0, 0],
                                 [0, 1, 0],
                                 [0, 0, 0]],
                                dtype=pars["dtype"]))
    return ham
```

FIG. 11. The function we used to express Hamiltonian. The matrix presents all the conditions when the neighbor spinors take different values.

```python
symmetry_bases["tricritical_ising"] = np.array([[        tri_2/3,  -(tri_1*tri_2)/3,  tri_1],
                                                [-(tri_1*tri_2)/3,         2*tri_2/3,     1],
                                                [           tri_1,                 1,     0]],
                                               dtype=np.complex_) /np.sqrt(3)
```

FIG. 12. The symmetry bases we chose to operate the tensor.

```python
def get_initial_tensor(pars, **kwargs):
    if kwargs:
        pars = pars.copy()
        pars.update(kwargs)
    model_name = pars["model"].strip().lower()
    ham = hamiltonians[model_name](pars)
    boltz = np.exp(-pars["beta"]*ham)
    T_0 = np.einsum('ab,bc,cd,da->abcd', boltz, boltz, boltz, boltz)
    if pars["symmetry_tensors"]:
        u = symmetry_bases[model_name]
        u_dg = u.T.conjugate()     #共轭转置
        T_0 = scon((T_0, u, u, u_dg, u_dg),
                   ([1,2,3,4], [-1,1], [-2,2], [3,-3], [4,-4]))
        cls, dim, qim = symmetry_classes_dims_qims[model_name]
        T_0 = cls.from_ndarray(T_0, shape=[dim]*4, qhape=[qim]*4,
                               dirs=[1,1,-1,-1])
    else:
        T_0 = Tensor.from_ndarray(T_0)
    return T_0
```

FIG. 13. The code used to generate the initial tensor, which is the basic element of the tensor network. Those integers are the numbers to denote the legs.

```python
tricritical_ising_0 = (0,(1,0,1,1,2,2,4,4,7,8,12,14,21,24,34,41))
tricritical_ising_716 = (7/16,(1,1,1,2,3,4,6,8,11,15,20,26,35,45,58,75))
tricritical_ising_110 = (1/10,(1,1,1,2,3,4,6,8,11,15,20,26,35,45,58,75))
tricritical_ising_35 = (3/5,(1,1,2,2,4,5,8,10,15,19,27,34,47,60,80,101))
tricritical_ising_380 = (3/80,(1,1,2,3,4,6,9,12,17,23,31,41,55,71,93,120))
tricritical_ising_32 = (3/2,(1,1,2,2,4,5,8,10,15,19,27,34,47,58,79,99))
```

FIG. 14. The minimal characters of the tricritical Ising model.

```python
elif modelname == "tricritical_ising" and not alpha:
#lead to identity
    if qnum is None or qnum == 0:
        prima_pairs += ((tricritical_ising_0, tricritical_ising_0),
                        (tricritical_ising_110, tricritical_ising_110),
                        (tricritical_ising_35, tricritical_ising_35),
                        (tricritical_ising_32, tricritical_ising_32))
    if qnum is None or qnum == 1:
        prima_pairs += ((tricritical_ising_380, tricritical_ising_380),
                        (tricritical_ising_716, tricritical_ising_716))
elif modelname == "tricritical_ising" and alpha and np.allclose(np.abs(alpha), np.pi):
    if qnum is None or qnum == 0:
        prima_pairs += ((tricritical_ising_380, tricritical_ising_380),
                        (tricritical_ising_716, tricritical_ising_716))
    if qnum is None or qnum == 1:
        prima_pairs += ((tricritical_ising_32, tricritical_ising_0),
                        (tricritical_ising_35, tricritical_ising_110),
                        (tricritical_ising_110, tricritical_ising_35),
                        (tricritical_ising_0, tricritical_ising_32))
```

FIG. 15. The code used to distinguish the identity and the $D_{\epsilon''}$ defect.

```python
def get_T(pars):
    T = tensordispenser.get_normalized_tensor(pars)
    parts = ()
    if pars["do_coarse_momenta"]:
        # TODO Should this bond dimension be an independent parameter?
        chis = [chi for chi in pars["chis_trg"]]
        T_orig = T
        if pars["print_errors"]:
            print("Building the coarse-grained transfer matrix times "
                  "translation.")
        T_dg = T.conjugate().transpose((0,3,2,1))
        y_env = scon((T, T, T_dg, T_dg),
                     ([1,-1,5,2], [5,-2,3,4], [1,2,6,-3], [6,4,3,-4]))
        U = y_env.eig((0,1), (2,3), hermitian=True, chis=chis)[1]
        y = U.conjugate().transpose((2,0,1))
        y_dg = y.conjugate().transpose((1,2,0))
        SW, NE = T.split((0,3), (1,2), chis=chis)
        SW = SW.transpose((0,2,1))
        NE = NE.transpose((1,2,0))
        T = scon((NE, y, T, SW, y_dg),
                 ([1,3,-1], [-2,1,4], [3,4,6,5], [6,-3,2], [5,2,-4]))
        parts = (y, y_dg, NE, SW, T_orig)
        # TODO quantify the error in this coarse-graining and print it.
    return T, parts
```

FIG. 16. Part of the code to get the tensor after coarse-graining.

```
Every AbelianTensor has the following attributes:

shape: A list of dims, one dim per leg. Every dim is a list of
integerers that are the dimensions of the different quantum number
blocks along that leg.

qhape: A list of qims, one qim per leg. Every qim is a list of
unique integers that are the quantum numbers (qnums) of that leg.
The quantum numbers are in one-to-one correspondence with the
elements of the dims, so that qhape[i][j] and shape[i][j] are the
qnum and dimension of the same block.

dirs: A list of integers -1 or 1, one for each leg. 1 means that the
corresponding leg is outgoing, -1 means incoming.

qodulus: An integer or None. If an integer, then all arithmetic on
the quantum numbers is done modulo qodulus. If None then arithmetic
on qnums is just usual integer arithmetic.

sects: A dict of numpy arrays, with combinations of quantum numbers
as keys. Every key must a tuple of quantum numbers, one for each
leg, and each one of them being from the qim of that leg. The value
of the dict at this key is the block (or "sector" or "sect")
corresponding to these quantum numbers. If the tensor is invariant
under a symmetry (see invar) then only certain blocks are allowed to
be set, but even in such a cause not all allowed blocks must be set.
For the treatement of unset blocks see defval.

dtype: A numpy dtype, that is the dtype of all the sects.
```

FIG. 17. Some annotation used to show the meaning of some parameters(qnum, shape, et al).