Real Analysis

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1 Introduction

Main goals:

- Measure theory: Extend the concept of length of intervals to general subsets of $\mathbb{R}$.
- Lebesgue integral: more convenient than the Riemann integral, in particular, for interchange of limit and integral.

1.1 The extended real numbers system

$$\overline{R} = \mathbb{R} \cup \{-\infty, \infty\}.$$

- ordering: $-\infty < x < \infty$ for all $x \in \mathbb{R}$.
- $\sup A$ and $\inf A$.
- algebraic operations involving $\pm\infty$: for addition and substraction, $\infty+(-\infty)$, $\infty - \infty$, $(-\infty) + \infty$, $(-\infty) - (-\infty)$ are not allowed; for multiplication, use $0 \cdot \infty = 0 \cdot (-\infty) = 0$.

1.2 Sum of series in $[0, \infty]$

Let $\{a_n\}_{n=1}^\infty$ be a sequence in $[0, \infty] =: \overline{R}_+$. If one of a_n's is ∞ or if all of them are in $\mathbb{R}$ but the series $\sum_{n=1}^\infty a_n$ diverges, we shall write

$$\sum_{n=1}^\infty a_n = \infty.$$

2 Sets

2.1 Axiom of Choice and Zorn's lemma

Axiom of Choice. *Let X be a set whose elements non-empty sets. Then there is a map defined on X such that for any $A \in X$, $f(A) \in A$.*

The following Zorn's Lemma is an equivalent statement of Axiom of Choice and sometimes more convenient to use.

Definition 2.1. *A* partial order *on a non-empty set X is a relation, often denoted by $\leq$, such that*

- *$a \leq a$ for each $a \in X$;*
- *For any $a, b, c \in X$, if $a \leq b$ and $b \leq c$ then $a \leq c$.*
- *if $a \leq b$ and $b \leq a$ then $a = b$.*

A subset E of X is totally ordered *if for any $a, b \in E$, either $a \leq b$ or $b \leq a$. A member x of X is called an* upper bound *for a subset E of X if for any $a \in E$, $a \leq x$ holds. A member x of X is called* maximal *if for any $x' \in X$, $x \leq x'$ does not hold.*

Zorn's Lemma. *Let X be a partially ordered non-empty set for which every totally ordered subset has an upper bound. Then X has a maximal element.*

Proposition 2.2. *Every vector space has a basis.*

Proof. Let V be a vector space over some field $\mathbb{F}$. Recall that a subset B of V is *linearly independent* if for any positive integer n, any $b_1, b_2, \ldots, b_n \in B$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$,

$$\lambda_1 b_2 + \lambda_2 b_2 + \cdots + \lambda_n b_n = 0 \Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

Let $\mathcal{B}$ denote the collection of all linear independent subsets of V and define a partial order $\leq$ as follows: $B_1 \leq B_2$ if $B_1 \subset B_2$.

Let us prove that any totally ordered subset $\mathcal{B}'$ of $\mathcal{B}$ has an upper bound. It suffices to show that $B' := \bigcup_{B \in \mathcal{B}'}$ is lineally independent so that it is a desired upper bound. To this end, let $b_1, \ldots, b_n \in B'$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ be such that $\lambda_1 b_1 + \cdots + \lambda_n b_n = 0$. we need to show that $\lambda_1 = \cdots = \lambda_n = 0$. Let $B_1, B_2, \ldots, B_n \in \mathcal{B}'$ such that $b_i \in B_i$ for each $1 \leq i \leq n$. Since $\mathcal{B}'$ is totally ordered, for any

$1 \le i, j \le n$ either $B_i \subset B_j$ or $B_j \subset B_i$. Thus there is $i_0 \in \{1, 2, \ldots, n\}$ such that $B_{i_0} = \bigcup_{i=1}^{n} B_i$. So $b_1, b_2, \ldots, b_n$ are contained in the linearly independent set B_{i_0}, which implies that $\lambda_1 = \cdots = \lambda_n = 0$.

By Zorn's Lemma, $\mathcal{B}$ has a maximal element B. Let us show that B is a basis of V. Otherwise, there is $v \in B$ which cannot be expressed as a linear combination of elements of B. Hence $B \cup \{v\}$ is again linearly independent, contradicting the maximality of B. □

2.2 Countable

Definition 2.3. *We say that two sets A and B are* equipotent *(or have the same* cardinality*) if there exists a bijection $f : A \to B$.*

Definition 2.4. *A set E is called* countable *if it is equipotent to a subset of $\mathbb{N} = \{1, 2, \ldots\}$. A set is called* uncountable *if it is not countable.*

In other words, a set E is countable if either it is finite or there exists a bijective $f : E \to \mathbb{N}$.

Proposition 2.5.

1. *If $\{E_n\}_{n=1}^{\infty}$ is a sequence of countable sets, then $\bigcup_{n=1}^{\infty} E_n$ is countable.*
2. *A subset of a countable set is countable.*
3. *The set of all rational numbers $\mathbb{Q}$ is countable.*

Proof. (1) If all these sets E_n are finite, then it is easy to show that $\bigcup_n E_n$ is finite or countably infinite. For the general case, list the elements of E_n by $a_{n,m}$, $m = 1, 2, \ldots$. Then for each $k \ge 1$, the set $F_k = \{a_{n,m} : n + m = k\}$ is finite. As $\bigcup_n E_n = \bigcup_k F_k$, the statement follows.

(2) Let E be a countable set and let $A \subset E$. If E is finite, then A is finite. Assume that there is a bijection $f : E \to \mathbb{N}$ and A is infinite. Then $f(A)$ is an infinite subset of $\mathbb{N}$. Define $n_1 = \inf f(A)$ and for each $k \ge 1$, define inductively $n_{k+1} = \inf f(A) \setminus \{n_1, n_2, \ldots, n_k\}$. Then $f(A) = \{n_1, n_2, \ldots\}$. Definie $g : f(A) \to \mathbb{N}$ by $n_k \mapsto k$. Then $g \circ f : A \to \mathbb{N}$ is a bijection.

(3) For each $n \ge 1$, let $E_n = \{m/n : m \in \mathbb{Z}\}$. Then E_n is countable. By (1), so is $\mathbb{Q} = \bigcup_n E_n$. □

Theorem 2.6. *The set $\mathbb{R}$ is uncountable.*

To prove the theorem, we shall need the following lemma, which is useful in verifying that two infinite sets have different cardinality.

Lemma 2.7. *Let Y be an arbitrary set and let $\mathcal{Y}$ denote the set of all subsets of Y. Then there does not exist a bijection from Y to $\mathcal{Y}$.*

Proof of Lemma 2.7. Arguing by contradiction, assume that there exists a bijection $\varphi : Y \to \mathcal{Y}$. Let

$$A = \{y \in Y : y \notin \varphi(y)\}.$$

Since φ is a bijection, there exists $a \in Y$ such that $\varphi(a) = A$. If $a \in A$, then by definition of the set A, we have $a \notin \varphi(a) = A$, a contradiction. If $a \in Y \setminus A$, then by definition of A, we have $a \in \varphi(a) = A$, again a contradiction! □

Proof of Theorem 2.6. Let $\mathcal{N}$ denote the set of all subsets of $\mathbb{N} = \{1, 2, \ldots\}$. Define a map $f : \mathcal{N} \to \mathbb{R}$ as follows: for each $A \subset \mathbb{N}$,

$$f(A) = \sum_{i=1}^{\infty} \frac{\chi_A(i)}{2^i}.$$

Let

$$\mathcal{N}_0 = \{A \subset \mathbb{N} : \text{ either } A \text{ or } \mathbb{N} \setminus A \text{ is finite}\}.$$

Then $\mathcal{N}_0$ is countable and thus by Lemma 2.7, $\mathcal{N} \setminus \mathcal{N}_0$ is uncountable. Note that f is injective on $\mathcal{N} \setminus \mathcal{N}_0$. It follows that there is a bijection from $U = f(\mathcal{N} \setminus \mathcal{N}_0) \subset \mathbb{R}$ onto a uncountable set. Therefore $\mathbb{R}$ is uncountable. □

Theorem 2.8 (Cantor-Schöder-Berstein)**.** *Let A, B be sets. Assume that there exist injections $f : A \to B$ and $g : B \to A$. Then A and B are equipotent.*

Proof. Without loss of generality, we may assume that $B \subset A$ and g is the inclusion map. (Otherwise, putting $\tilde{B} = g(B)$ and $\tilde{f} = g \circ f$, then $\tilde{f}$ is an injective from A to its subset $\tilde{B}$ and we only need to show that A and $\tilde{B}$ are equipotent.) Let $C_0 = A \setminus B$ and $C_n = f^n(C_0)$ for each $n \geq 1$. Let $D = B \setminus (\bigcup_{n=1}^{\infty} C_n)$. Then A is the disjoint union of D and C_n, $n \geq 0$, while B is the disjoint union of D and C_n, $n \geq 1$. Define

$$F(x) = \begin{cases} x & \text{if } x \in D, \\ f(x) & \text{if } x \in C_n \text{ for some } n \geq 0. \end{cases}$$

Then F is a bijection from A to B. □

Theorem 2.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Then f is continuous except possibly at countably many points.*

Proof. Without loss of generality, we assume that f is increasing. For each $x \in \mathbb{R}$, let

$$\delta(x) = \lim_{y \searrow x} f(y) - \lim_{y \nearrow x} f(y) \geq 0.$$

Then f is continuous at x if and only if $\delta(x) = 0$. In order to show that

$$E := \{x \in \mathbb{R} : f \text{ is not continuous at } x\} = \{x \in \mathbb{R} : \delta(x) > 0\}$$

is countable, it suffices to show that for all positive integers n, k,

$$E_{n,k} = \{x \in (-n, n) : \delta(x) > k^{-1}\}$$

is finite. To this end, we shall prove that

$$\# E_{n,k} < k(f(n) - f(-n)). \tag{1}$$

Otherwise, there exists $-n = x_0 < x_1 < x_2 < \cdots < x_m < x_{m+1} = n$ with $m \geq k(f(n) - f(-n))$ and $\delta(x_m) > 1/k$. By definition of $\delta(x)$,

$$x_0 < a_1 < x_1 < b_1 < a_2 < x_2 < b_2 < \cdots < a_m < x_m < b_m < x_{m+1}$$

such that

$$f(b_i) - f(a_i) > 1/k.$$

Then

$$f(n) - f(-n) = \sum_{i=0}^{m} (f(x_{i+1}) - f(x_i)) \geq \sum_{i=1}^{m} (f(b_i) - f(a_i)) > m/k,$$

which implies that $m < k(f(n) - f(-n))$, a contradiction! $\square$

An alternative proof. Without loss of generality, we assume that f is increasing. Let $E = \{x \in \mathbb{R} : f \text{ is not continuous at } x\}$. Then for each $x \in E$,

$$a_x := \lim_{y \nearrow x} f(y) < b_x := \lim_{y \searrow x} f(y).$$

Since f is monotone, the intervals (a_x, b_x) are pairwise disjoint. For each $x \in E$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q(x) \in (a_x, b_x)$. This defines a function $q : E \to \mathbb{Q}$. Since $q(x_1) \neq q(x_2)$ for any $x_1, x_2 \in E$ with $x_1 \neq x_2$, q is injective. Since $\mathbb{Q}$ is countable, E is countable. $\square$