# Potential automorphy for $GL_n$

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**Abstract** We prove potential automorphy results for a single Galois representation $G_F \to GL_n(\overline{\mathbb{Q}}_l)$ where $F$ is a CM number field. The strategy is to use the $p, q$ switch trick to go between the $p$-adic and $q$-adic realisation of a certain variant of the Dwork motive. We choose this variant to break self-duality shape of the motives, but not the Hodge-Tate weights. Another key result to prove is that certain $p$-adic representations we choose that come from the Dwork motives is ordinarily automorphic. One input is the automorphy lifting theorem in Allen et al.: (Potential automorphy over CM fields, Cornell University, New York 2018) .

## 1 Introduction

In this paper we prove potential automorphy theorems for $n$-dimensional $l$-adic and residual representations of the absolute Galois group of an imaginary CM field.

The precise statement of the theorem for residual representations is as following.

**Theorem 1.1** *Suppose $F$ is a CM number field, $F^{\mathrm{av}}$ is a finite extension of $F$ and $n \geq 2$ is a positive integer. Let $l$ be an odd prime number and suppose that*

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$$\bar{r} : \mathrm{Gal}(\overline{F}/F) \to GL_n(\mathbb{F}_{l^s})$$

*is a continuous semisimple representation. Then there exists a finite CM Galois extension $F'/F$ linearly disjoint from $F^{\mathrm{av}}$ over $F$ such that $\bar{r}\,|_{\mathrm{Gal}(\overline{F}/F')}$ is ordinarily automorphic.*

We first breifly recall the definitions of the terms appearing in the theorem.

Recall that for $E$ any CM (or totally real) field, we could attach to any regular algebraic cuspidal automorphic representation $\pi$ of $GL_n(\mathbb{A}_E)$ an $l$-adic Galois representation of $G_E$ satisfying certain local-global compatibility condition by the main theorem of [9].

More precisely, fix an isomorphism $\overline{\mathbb{Q}}_l \to \mathbb{C}$. For such a $\pi$, there is a unique continuous semisimple representation

$$r_{l,\iota}(\pi) : G_E \to GL_n(\overline{\mathbb{Q}}_l)$$

such that, if $p \neq l$ is a rational prime above which $\pi$ and $E$ are unramified and if $v|p$ is a prime of $E$, then $r_{l,\iota}(\pi)$ is unramified at $v$ and

$$r_{l,\iota}(\pi)|^{ss}_{W_{E_v}} = \iota^{-1} rec_{E_v}(\pi_v |\det|_v^{(1-n)/2})$$

here $rec_{E_v}$ denotes the local Langlands correspondance for $E_v$ and $|^{ss}$ denotes the semisimplification.

**Definition 1.2** For a $p$-adic local field $L$ and a continuous representation $\rho : G_L \to GL_n(\overline{\mathbb{Q}}_p)$, we say it is ordinary with regular Hodge-Tate weight if there exists a weight $\lambda = (\lambda_{\tau,i}) \in (\{(a_1, \ldots, a_n)|a_1 \geq \cdots \geq a_n\})^{\mathrm{Hom}(L, \overline{\mathbb{Q}}_p)} =: (\mathbb{Z}_+^n)^{\mathrm{Hom}(L, \overline{\mathbb{Q}}_p)}$ such that there is an isomorphism:

$$\rho \sim \begin{pmatrix} \psi_1 & * & * & * \\ 0 & \psi_2 & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \psi_n \end{pmatrix},$$

where for each $i = 1, \ldots, n$ the character $\psi_i : G_L \to \overline{\mathbb{Q}}_p^\times$ agrees with the character

$$\sigma \in I_L \mapsto \prod_{\tau \in \mathrm{Hom}(L, \overline{\mathbb{Q}}_p)} \tau(\mathrm{Art}_L^{-1}(\sigma))^{-(\lambda_{\tau,n-i+1}+i-1)}$$

on an open subgroup of the inertia group $I_L$.

**Definition 1.3** For a Galois representation $r : G_E \to GL_n(\overline{\mathbb{Q}}_l)$, we say it is automorphic if there exists a regular algebraic cuspidal automorphic representation $\pi$ such that $r \cong r_{l,\iota}(\pi)$. And for a residual representation $\overline{r} : G_E \to GL_n(\overline{\mathbb{F}}_l)$, we say it is automorphic if there exists a lift $r$ of $\overline{r}$ that is automorphic. We say it is $\iota$-ordinarily automorphic if there exists an automorphic lift $r \cong r_{l,\iota}(\pi)$ such that the automorphic representation $\pi$ it corresponds to is $\iota$-ordinary (in the sense of [8] Definition 5.3) at all places $v$ above $l$.

We also remark here that restricting to some $G_{F'}$ for a Galois extension $F'/F$ that avoids a prescribed finite extension $F^{av}$ of $F$ can ensure that the image $\overline{r}(G_{F'})$ does not shrink.

Combine our main theorem for residual representation Theorem 1.1 with the automorphy lifting theorem 6.1.2 from [1] and the main result of [14], we obtain a potential automorphy theorem for a single $l$-adic Galois representation into $GL_n$.

**Theorem 1.4** *Suppose $F$ is a CM number field, $F^{av}$ is a finite extension of $F$ and $n \geq 2$ is a positive integer. Let $l$ be an odd prime number. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ and suppose that*

$$r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$$

*is a continuous representation satisfying the following condition:*

- *$r$ is unramified almost everywhere.*
- *For each place $v|l$ of $F$, the representation $r|_{G_{F_v}}$ is potentially semistable, ordinary with regular Hodge-Tate weights.*
- *$\overline{r}$ is absolutely irreducible and decomposed generic (See [1] Definition 4.3.1). The image of $\overline{r}|_{G_{F(\zeta_l)}}$ is enormous (See [1] Definition 6.2.28).*
- *There exists $\sigma \in G_F - G_{F(\zeta_l)}$ such that $\overline{r}(\sigma)$ is a scalar.*

*Then there exists a finite CM Galois extension $F'/F$ linearly disjoint from $F^{av}$ over $F$ such that $r \mid_{G_{F'}}$ is ordinarily automorphic.*

Previously there are potential automorphy results for $\overline{r}$ and $r$ that take value in $GSp_{2n}$ ( [10]) or more generally, the subgroup of $GL_n$ that preserves a nondegenerate form up to a scalar ( [4]). The strategy of proving the main theorems in this paper is based on the strategy of proving theorems in these paper. But there are also many crucial differences.

The main idea of the proof of Theorem 1.1 is as the following. The prime $l$ is given. But we will choose some positive integer $N$ and another prime $l'$ with good properties. Note that this choice make certain arguments for the $l'$-related objects easier than their $l$-related counterpart.

Consider the Dwork family $Y \subset \mathbb{P}^{N-1} \times (\mathbb{P}^1 \backslash (\mu_N \cup \{\infty\}))$ defined by the following equation:

$$X_1^N + X_2^N + \; + X_N^N = Nt X_1 X_2 \ldots X_N.$$

for the good $N$ we will choose. The variety comes equipped with an action of the group

$$H_0 := \{(\xi_1, \ldots, \xi_N) \in \mu_N^N : \xi_1 \cdots \xi_N = 1\}/\mu_N.$$

(see Sect. 3). After picking a character $\chi$ of $H_0$, we may consider the motives such that their $l$(or $l'$)-adic realisation is the $\chi$-eigenspace of the $(N-2)$-th(middle degree) etale cohomology of any fibre of this family with coefficients $\overline{\mathbb{Q}}_l$ (or $\overline{\mathbb{Q}}_{l'}$). We will denote such $l$(or $l'$)-adic cohomology of the fibre over the point $t$ as $V_{\lambda,t}$(or $V_{\lambda',t}$), where $\lambda, \lambda'$ is a place of $\mathbb{Q}(\zeta_N)$ above $l, l'$. Note that here we will choose a $\chi$ that is of a shape artificially made to break the self-duality of the motive. However, the self-duality shape of the Hodge-Tate weights will be preserved. In fact, they are a string of consecutive integers. We will try to find a point $t$ on the base defined over an extension field $F'$, such that the mod-$l$ residual Galois representation $V[\lambda]_t$ given by the fibre of the motive over $t$ is isomorphic to the $\bar{r}$ in the theorem, while the mod-$l'$ residual Galois representation $V[\lambda']_t$ given by the fibre of the motive over $t$ is isomorphic to $\bar{r}_{l',\iota}(\pi_1)$ for some known ordinarily automorphic representation $r_{l',\iota}(\pi_1)$, both as representation of $G_{F'}$. If such a point exists, then we could apply ordinary automorphy lifting theorem 6.1.2 of [1] to see $V_{\lambda',t}$ is automorphic, and conclude that $V_{\lambda,t}$ is automorphic. Hence $\bar{r}$ is automorphic.

The above is a very rough summary of what we did in this paper. Let us be more precise now.

The first problem that is worth more explanantion is the existence of such a point $t$. Assume $\bar{r}$ and $\bar{r}_{l',\iota}(\pi)$ can be defined as representation over $k(\lambda)$ and $k(\lambda')$, where these are the residue fields for the places $\lambda$ and $\lambda'$ of $\mathbb{Z}[\zeta_N]$. The existence of such a point $t$ is guaranteed by a careful study of the moduli scheme that detects the isomorphisms between $\bar{r} \times \bar{r}_{l',\iota}(\pi)$ and the varying $V[\lambda]_t \times V[\lambda']_t$ as representation over $k(\lambda) \times k(\lambda')$, such that the top wedge of the isomorphism is fixed to be an a priori choice. Now the main property we use to prove the existence of such a point $t$ is the geometric connectivity of the moduli variety. And the geomoretic connectivity is in turn deduced from the result that the geometric monodromy map of this family surjects onto $SL_n(k(\lambda)) \times SL_n(k(\lambda'))$, over which the fiber over $t$ of the moduli scheme is a torsor. The proof of this surjectivity result involves combinatorial arguments that precisely make use of the shape of the charater $\chi$ we choose. In contrast, we know that if we had chosen the $\chi$ to be of some nice self-dual form, then the image of the geometric monodromy map would be contained

in some symplectic group $Sp_n(k(\lambda)) \times Sp_n(k(\lambda'))$. We remark that in previous work, other authors have considered the moduli variety parametrizing similar isomorphisms but with the condition that certain alternating forms on both representation spaces need to be preserved, where the alternating form on the varying cohomology is induced by Poincare duality and the self-dual shape of the $\chi$ they chose.

In the above procedure, after showing that the geometric monodromy has image in $SL_n(k(\lambda)) \times SL_n(k(\lambda'))$ (using the shape of $\chi$), we see that the spaces $\wedge^n(V[\lambda]_t \times V[\lambda']_t)$ as characters of $G_F$ does not depend on the base point $t$. Thus to construct the moduli scheme, it suffices to construct a fixed isomorphism between $\wedge^n(\bar{r} \times \bar{r}_{l',\iota}(\pi))$ and $\wedge^n(V[\lambda]_t \times V[\lambda']_t)$ for any chosen $t$ an $F$ point of the base. However, this isomorphism does not a priori exist. We get around this problem by restricting to a smaller $G_{F'}$ and twisting $\bar{r} \times \bar{r}_{l',\iota}(\pi)$ by a character $\overline{\chi}_1 \times \overline{\chi}_2 : G_{F'} \to k(\lambda)^\times \times k(\lambda')^\times$, so that we would like to construct the moduli scheme as the one detecting isomorphisms between $(\bar{r} \times \bar{r}_{l',\iota}(\pi)) \otimes (\overline{\chi}_1 \times \overline{\chi}_2)$ and $V[\lambda]_t \times V[\lambda']_t$. Note that we need $\overline{\chi}_1 \times \overline{\chi}_2$ to take value in exactly $k(\lambda)^\times \times k(\lambda')^\times$ because we want the fiber of the moduli scheme to be a torsor under the image $SL_n(k(\lambda)) \times SL_n(k(\lambda'))$ of the geometric monodromy map because this is crucial to show the geometric connectivity of the moduli scheme.

Now choosing $t = 0$, to construct an isomorphism between $\det((\bar{r} \times \bar{r}_{l',\iota}(\pi)) \otimes (\overline{\chi}_1 \times \overline{\chi}_2))$ and $\det(V[\lambda]_t \times V[\lambda']_t)$, amounts to taking an "$n$-th root" of the character $(\det V[\lambda]_0 \times V[\lambda']_0)^{-1} \otimes \det(\bar{r} \times \bar{r}_{l',\iota}(\pi))$ as character valued in $k(\lambda)^\times \times k(\lambda')^\times$, where $V[\lambda]_0, V[\lambda']_0$ denotes the mod $l, l'$ cohomology of the fibre over 0. The first step to make this adjustment work is that we need

- $(\det V[\lambda]_0 \times V[\lambda']_0)^{-1} \otimes \det(\bar{r} \times \bar{r}_{l',\iota}(\pi))$ has image in $(k(\lambda)^\times)^n \times (k(\lambda')^\times)^n$ as a $G_{F'}$ representation.

We remark that this condition is proved by a computation for the fibre over 0, where there is a good description. The computation is done in 3.9. Then, we will use Lemma 2.1 to deduce that this condition above enables us to construct such an "$n$-th root" of character while also making sure that $F'$ satisfies certain linearly disjoint properties.

The second problem is that to apply ordinary automorphy lifting theorems, we also need to show $V_{\lambda',t}$ is ordinary (as $G_{F'_{v'}}$ representation where $v'$ is any $l'$-adic places of $F'$) and $V_{\lambda,t}$ (as a $G_{F'_v}$ representation where $v$ is any $l$-adic place of $F'$) is ordinarily automorphic.

The proof of $V_{\lambda',t}$ being ordinary is relatively easy. We just pick $t \in \mathbb{A}^1(F')$ that is $l'$-adicly close to 0. Applying 2.4, we may check ordinarity via an examination of $D_{\mathrm{cris}}(V_{\lambda',t})$. The comparison theorem identifies $D_{\mathrm{cris}}(V_{\lambda',t})$

and $D_{\mathrm{cris}}(V_{\lambda',0})$, and hence reduces the proof to the case $t = 0$. In that case, $V_{\lambda',0}$ actually splits into characters as a $G_{F'}$ representation.

To prove that $V_{\lambda,t}$ is $\iota$-ordinarily automorphic (with the assumption that $V_{\lambda',t}$ is $\iota'$-automorphic and $V_{\lambda',t} = r_{l',\iota'}(\pi)$) is harder and relies on a maximal unipotence result of the monodromy of $V_{\lambda',t}$, as well as Varma's local-global compatibility result as in [17]. The idea is to choose $t \in \mathbb{A}^1(F')$ that has $l$-adic valuation $< 0$. Then an argument similar to that of Lemma 1.15 part 2 of [10] shows that the nilpotent operator $N$ in the Weil-Deligne representation associated to $V_{\lambda',t}$ (as a $G_{F'_v}$ representation, where $v$ is an $l$-adic place of $F'$) is maximally nilpotent. Then the result of [17] shows that $\pi_v$ is Steinberg. Hence $\pi_v$ being $\iota$-ordinary follows from a variant of Lemma 5.6 of [8].

The difference between the case of $l$ and $l'$ arises partially from the fact that $l$ is given but we may choose $l'$ arbitrarily.

Lastly the $\pi_1$ we use such that $r_{l',\iota'}(\pi_1)$ is ordinarily automorphic and lift $V[\lambda']_t$ is such that $r_{l',\iota'}(\pi_1)$ is a symmetric tensor power of the Tate module of an elliptic curve over $\mathbb{Q}$.

With the above input and an awkward choice of the character $\chi$ of $H_0$, plus several technical algebraic number theory lemmas listed in Sect. 2, we will finally prove the main theorems in Sect. 4.

## 2 Several lemmas

In this section we prove several technical lemmas and fix some notation that will appear later in the paper. Throughout this section, $l$ is an odd prime.

Let us first state the properties we will use throughout the paper regarding the notion of linearly disjoint fields.

- If $A$ and $B$ are extensions of $C$ then $A$, $B$ linearly disjoint over $C$ implies $A \cap B = C$, and the converse is true if $A$ or $B$ is finite Galois over $C$.
- If $A \supset B \supset C$ and $D \supset C$ with $A$ and $D$ linearly disjoint over $C$, then $A$ and $BD$ are linearly disjoint over $B$. In particular $A \cap BD = B$.

**Lemma 2.1** *For a CM field $M$, a finite Galois extension $F_0/\mathbb{Q}$, a finite field $\mathbb{F}_{l^r}$ containing all $n$-th roots of unity, and a character $\chi : G_M \to (\mathbb{F}_{l^r}^\times)^n$, there exists a finite totally real Galois extension $L/\mathbb{Q}$ linearly disjoint with $F_0$ over $\mathbb{Q}$ and such that if we denote $F_1 = LM$, there exists a character $\psi : G_{F_1} \to \mathbb{F}_{l^r}^\times$ such that $\psi^n = \chi|_{G_{F_1}}$.*

*Proof* Consider the long exact sequence associated to the following short exact sequence of $G_M$-module with trivial action:

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{F}_{l^r}^\times \xrightarrow{(\cdot)^n} (\mathbb{F}_{l^r}^\times)^m \longrightarrow 0,$$

where we write $n = l^a m$, $l \nmid m$, we have:

$$H^1(G_M, \mathbb{F}_{l^r}^\times) \xrightarrow{(\cdot)^n} H^1(G_M, (\mathbb{F}_{l^r}^\times)^m) \xrightarrow{\delta} H^2(G_M, \mathbb{Z}/m\mathbb{Z}).$$

Now $\chi \in H^1(G_M, (\mathbb{F}_{l^r}^\times)^m)$. If we let $\tilde{\chi} = \delta(\chi)$, we are reduced to finding a Galois CM extension $F_1 \supset M$ of the form $F_1 = LM$ for some $L$ linearly disjoint with $F_0$ over $\mathbb{Q}$ such that the obstruction $\tilde{\chi}$ is killed by the restriction map $H^2(G_M, \mathbb{Z}/m\mathbb{Z}) \to H^2(G_{F_1}, \mathbb{Z}/m\mathbb{Z})$.

Consider the map $H^2(G_M, \mathbb{Z}/m\mathbb{Z}) \to \prod_v H^2(G_{M_v}, \mathbb{Z}/m\mathbb{Z})$ given by restriction. The image actually lands in $\bigoplus_v H^2(G_{M_v}, \mathbb{Z}/m\mathbb{Z})$ because any element in $H^2(G_M, \mathbb{Z}/m\mathbb{Z})$ is inflated from some $\phi \in H^2(\text{Gal}(M'/M), \mathbb{Z}/m\mathbb{Z})$ for some $M'/M$ a finite extension and for those primes $v$ of $M$ that is unramified in $M'$, the image of $\phi$ in $H^2(G_{M_v}, \mathbb{Z}/m\mathbb{Z})$ by restriction actually lands in $H^2(\text{Gal}(M_v^{\text{nr}}/M_v), \mathbb{Z}/m\mathbb{Z})$, which is 0 since the cohomological dimension of $\hat{\mathbb{Z}}$ is 1.

The first step is to take an CM extension $F_2/M$ that is of the form $L_2 M$ for a totally real $L_2$ Galois over $\mathbb{Q}$ that is linearly disjoint with $F_0$ over $\mathbb{Q}$, such that in the following commutative diagram, the image of $\tilde{\chi}$ in the upper right corner is 0:

$$
\begin{array}{ccc}
H^2(G_{F_2}, \mathbb{Z}/m\mathbb{Z}) & \longrightarrow & \bigoplus_w H^2(G_{F_{2,w}}, \mathbb{Z}/m\mathbb{Z}) \\
\uparrow & & \uparrow \\
H^2(G_M, \mathbb{Z}/m\mathbb{Z}) & \longrightarrow & \bigoplus_v H^2(G_{M_v}, \mathbb{Z}/m\mathbb{Z})
\end{array}
$$

Let $\bigoplus \tilde{\chi}_v$ be the image of $\tilde{\chi}$ in $\bigoplus_v H^2(G_{M_v}, \mathbb{Z}/m\mathbb{Z})$. If we can take a CM extension $F_2/M$ of the above form such that for any $v$ with $\tilde{\chi}_v \neq 0$ and $w|v$ a place of $F_2$, $\zeta_m \in F_{2,w}$ and $m|[F_{2,w} : M_v(\zeta_m)]$, then the image of $\tilde{\chi}_v$ restricting to $H^2(G_{F_{2,w}}, \mathbb{Z}/m\mathbb{Z})$ is 0, since $H^2(G_{M_v(\zeta_m)}, \mathbb{Z}/m\mathbb{Z}) \cong H^2(G_{M_v(\zeta_m)}, \mu_m) \cong \frac{1}{m}\mathbb{Z}/\mathbb{Z}$, and the restriction map $\frac{1}{m}\mathbb{Z}/\mathbb{Z} \cong H^2(G_{M_v(\zeta_m)}, \mu_m) \to H^2(G_{F_{2,w}}, \mu_m) \cong \frac{1}{m}\mathbb{Z}/\mathbb{Z}$ is multiplication by $[F_{2,w} : M_v(\zeta_m)]$.

We can construct such an extension $F_2/M$ coming from $L_2/\mathbb{Q}$ linearly disjoint with $F_0$ over $\mathbb{Q}$ with prescribed local behavior for a finite number of primes $v$ of $M$ as the following:

Let $S_1$ be the set of rational primes lying under the primes $v$ of $M$ such that $\tilde{\chi}_v \neq 0$. Let $S_2 = \{\infty\}$ and $S = S_1 \cup S_2$. For each $q \in S_1$, let $M_q$ denote the composite of the image of all embeddings $\tau : M \hookrightarrow \overline{\mathbb{Q}}_q$. We fix an extension $E_q/M_q(\zeta_m)$ of order divisible by $m$ and Galois over $\mathbb{Q}_q$. Now we apply Lemma 4.1.2 of [5] to $F_0/\mathbb{Q}$ and the set of primes $S$ with prescribed local behavior:

- $E_q/\mathbb{Q}_q$ for all $q \in S_1$
- Trivial extension $\mathbb{R}/\mathbb{R}$ for $\infty \in S_2$

We get a finite totally real Galois extension $L_2/\mathbb{Q}$ that is linearly disjoint with $F_0$ over $\mathbb{Q}$ and that for any $w$ a place of $L_2$ over a $q \in S_1$, $(L_2)_w \cong E_q$ we defined above over $\mathbb{Q}_q$. Take $F_2 = ML_2$. For any prime $v$ of the field $M$ with $\tilde{\chi}_v \neq 0$ and $v'$ a prime of $F_2$ over $v$, then $q = v'_{|\mathbb{Q}} \in S_1$, thus $F_{2,v'} \supset (L_2)_{v'|_{L_2}} \supset E_q$ and so $\mathrm{Gal}(F_{2,v'}/M_v(\zeta_m))$ is of order divisible by $m$. So we have constructed the desired $L_2$ and $F_2$.

Now for any number field $F$ and $G_F$ module $A$, let $\mathrm{III}^i(F, A)$ be

$$\ker(H^i(G_F, A) \to \prod_v H^i(G_{F_v}, A))$$

where the product is over all places $v$ of $F$ (so is every product that follows).

Thus the first step yields a finite CM Galois extension $F_2/M$ that comes from some $L_2/\mathbb{Q}$ as described above such that the image $\tilde{\chi}_1$ of $\tilde{\chi}$ in $H^2(G_{F_2}, \mathbb{Z}/m\mathbb{Z})$ actually lies in $\mathrm{III}^2(F_2, \mathbb{Z}/m\mathbb{Z})$ .

The second step is to analyze $\mathrm{III}^2(F_2, \mathbb{Z}/m\mathbb{Z})$ and kill it after some further CM extension $F_1/F_2$ where $F_1 = L_1L_2M$, with $L_1$ totally real Galois over $\mathbb{Q}$ that would be specified later and such that $L := L_1L_2$ is linearly disjoint with $F_0$ over $\mathbb{Q}$.

Poitou–Tate duality(cf. [13] Theorem 8.6.7) gives a perfect pairing

$$\langle \cdot, \, \cdot \rangle : \mathrm{III}^2(F_2, \mathbb{Z}/m\mathbb{Z}) \times \mathrm{III}^1(F_2, \mu_m) \to \mathbb{Q}/\mathbb{Z}$$

satisfying the following compatibility for any finite extension $F_1/F_2$ and $x \in \mathrm{III}^1(F_1, \mu_m)$, $y \in \mathrm{III}^2(F_2, \mathbb{Z}/m\mathbb{Z})$ :

$$\langle x, \mathrm{Res}(y) \rangle = \langle \mathrm{Cor}(x), y \rangle$$

So we now choose such an extension $F_1/F_2$ such that $\mathrm{Cor}(x) = 0$, $\forall x \in \mathrm{III}^1(F_1, \mu_m)$, then by the perfectness, $\mathrm{Res}(y) = 0$, $\forall y \in \mathrm{III}^2(F_2, \mathbb{Z}/m\mathbb{Z})$.

Write $m = 2^r \prod_{i=1}^s p_i^{r_i}$. Decompose $\mathrm{III}^1(F_2, \mu_m) = \mathrm{III}^1(F_2, \mu_{2^r}) \times \prod_{i=1}^s \mathrm{III}^1(F_2, \mu_{p_i^{r_i}})$. The following lemma is basically Theorem 9.1.9 of [12].

**Lemma 2.2** *For any number field $F$, $\mathrm{III}^1(F, \mu_{p^r}) = 0$ or $\mathbb{Z}/2\mathbb{Z}$. The later case could happen only when $p = 2$.*

*In any case, $\mathrm{III}^1(F, \mu_{p^r}) = \mathrm{III}^1(F(\mu_{p^r})/F, \mu_{p^r})$ (defined in the proof) .*

*Proof* Set $K = F(\mu_{p^r})$. We have the following commutative diagram where each row and column (except the left column) are exact:

$$
\begin{array}{ccccc}
0 & \longrightarrow & H^1(G_K, \mu_{p^r}) & \longrightarrow & \prod_w H^1(G_{K_\omega}, \mu_{p^r}) \\
& & \uparrow & & \uparrow \\
0 \longrightarrow \mathrm{III}^1(F, \mu_{p^r}) & \longrightarrow & H^1(G_F, \mu_{p^r}) & \longrightarrow & \prod_v H^1(G_{F_v}, \mu_{p^r}) \\
& & \updownarrow & & \updownarrow \\
0 \longrightarrow \mathrm{III}^1(K/F, \mu_{p^r}) & \longrightarrow & H^1(\mathrm{Gal}(K/F), \mu_{p^r}) & \longrightarrow & \prod_v H^1(\mathrm{Gal}(K_w/F_v), \mu_{p^r})
\end{array}
$$

where $\mathrm{III}^1(K/F, \mu_{p^r})$ is defined by the exactness of the bottom row and the top row is exact because an element in the kernel corresponds to a cyclic extension of $K$ of order dividing $p^r$ that splits at all primes $w$ of $K$, which has to be trivial. (Again $\mu_{p^r} = \mathbb{Z}/p^r\mathbb{Z}$ as a $G_K$ module and $H^1$ is just Hom.)

A diagram chasing gives that $\mathrm{III}^1(F, \mu_{p^r}) = \mathrm{III}^1(K/F, \mu_{p^r})$. By Proposition 9.1.6 of [12], $H^1(\mathrm{Gal}(K/F), \mu_{p^r}) = 0$ except when

- $p = 2, r \geq 2$
- and $-1$ is in the image of $\mathrm{Gal}(K/F) \to (\mathbb{Z}/2^r\mathbb{Z})^\times$

In this case, $H^1(\mathrm{Gal}(K/F), \mu_{2^r}) = \mathbb{Z}/2\mathbb{Z}$. As a subspace of $H^1$ $(\mathrm{Gal}(K/F), \mu_{2^r})$, $\mathrm{III}^1(F, \mu_{p^r}) = \mathrm{III}^1(K/F, \mu_{p^r}) = 0$ or $\mathbb{Z}/2\mathbb{Z}$. $\qquad\square$

Recall the relation $\mathrm{Cor} \circ \mathrm{Res} = [F_1 : F_2]$ and the commutative diagram:

$$
\begin{array}{ccc}
H^1(G_{F_2}, \mu_{2^r}) & \xrightarrow{\quad \mathrm{Res} \quad} & H^1(G_{F_1}, \mu_{2^r}) \\
\uparrow & & \uparrow \\
H^1(\mathrm{Gal}(F_2(\mu_{2^r})/F_2), \mu_{2^r}) & \xrightarrow{\quad \mathrm{Res} \quad} & H^1(\mathrm{Gal}(F_1(\mu_{2^r})/F_1), \mu_{2^r})
\end{array}
$$

The bottom row is an isomorphism if we pick $F_1$ linearly disjoint with $F_2(\mu_{2^r})$ over $F_2$. If this is the case and $2 \mid [F_1 : F_2]$, then by Lemma 2.2

$$
\begin{aligned}
\mathrm{Cor}(\mathrm{III}^1(F_1, \mu_m)) &= \mathrm{Cor}(\mathrm{III}^1(F_1, \mu_{2^r})) \\
&= \mathrm{Cor}(\mathrm{III}^1(F_1(\mu_{2^r})/F_1, \mu_{2^r})) \\
&\subset \mathrm{Cor}(H^1(\mathrm{Gal}(F_1(\mu_{2^r})/F_1), \mu_{2^r})) \\
&= \mathrm{Cor}(\mathrm{Res}(H^1(\mathrm{Gal}(F_2(\mu_{2^r})/F_2), \mu_{2^r}))) \\
&= [F_1 : F_2] \cdot H^1(\mathrm{Gal}(F_2(\mu_{2^r})/F_2), \mu_{2^r}) \\
&= 0.
\end{aligned}
\tag{2.1}
$$

Here when we apply Cor to some group, we always mean Cor applied to the image of this group in $H^1(G_{F_1}, \mu_{2^r})$.

Now we construct an $L_1$ such that the associated extension $F_1/F_2$ satisfies the property that $F_1$ is linearly disjoint with $F_2(\mu_{2^r})$ over $F_2$ and $2 \mid [F_1 : F_2]$. Choose a rational prime $p$ and a local field $E_p$ Galois over $\mathbb{Q}_p$ that contains $M'_p$, the composite of the image of all embeddings $\tau : F_2 \hookrightarrow \overline{\mathbb{Q}}_p$, and is of order divisible by 2 over it. We again apply Lemma 4.1.2 of [5] to the extension $F_0 F_2(\mu_{2^r})/\mathbb{Q}$ and the set of primes $S$ consisting of $p$ and $\infty$ with prescribed local behavior:

- $E_p/\mathbb{Q}_p$
- Trivial extension $\mathbb{R}/\mathbb{R}$ for the place $\infty$.

We get a totally real Galois extension $L_1/\mathbb{Q}$ linearly disjoint with $F_0 F_2(\mu_{2^r})$ over $\mathbb{Q}$. The associated $F_1 = F_2 L_1 = M L_2 L_1$. Then $2 \mid [F_1 : F_2]$ because for any place $v$ of $F_1$ above $p$, $(F_1)_v \supset (L_1)_{v|_{L_1}} \cong E_p \supset (F_2)_{v|_{F_2}}$ and the last inclusion is of order divisible by 2. The property that $F_1 = F_2 L_1$ and $F_2(\mu_{2^r})$ are linearly disjoint over $F_2$ follows from the fact that $L_1$ and $F_2(\mu_{2^r})$ are linearly disjoint over $\mathbb{Q}$. Now $L_1$ and $F_0 F_2$ are linearly disjoint over $\mathbb{Q}$ implies that $L_1 L_2$ and $F_0 F_2$ are linearly disjoint over $L_2$. Hence $L_1 L_2 \cap F_0 = L_1 L_2 \cap F_0 F_2 \cap F_0 = L_2 \cap F_0 = \mathbb{Q}$.

We conclude that the image of $\tilde{\chi}$ in $H^2(G_{F_1}, \mathbb{Z}/m\mathbb{Z})$ is 0 by (2.1) and thus we can take an $n$-th root of $\chi|_{G_{F_1}}$ for $F_1 = LM \supset M$ and $L = L_1 L_2$ we constructed above finite totally real Galois over $\mathbb{Q}$ and linearly disjoint with $F_0$ over $\mathbb{Q}$.

$\square$

**Lemma 2.3** *Let $l$ be a rational prime. Given any positive integer $s$ and a finite set of rational primes $S$, we can find a positive integer $N$ not divisible by any primes in $S$ and $l$, and satisfying :*

- *Let $r$ be the smallest positive integer such that $N \mid l^r - 1$, then $s \mid r$.*
- *When $r$ is even, $N \nmid l^{r/2} + 1$*

*Proof* Factorize $s$ as $s = 2^{a_0} \prod_{i=1}^{m} p_i^{a_i}$. View $p_0 = 2$. We will construct a sequence of pairwise coprime integers $M_i$ (not divisible by any rational prime in $S$) and a sequence of integers $t_i$ with $t_i \geq a_i$ for $i = 0, 1, \ldots, m$, such that $M_i \mid l^r - 1$ if and only if $p_i^{t_i} \mid r$. Set $N = \prod_{i=0}^{m} M_i$ and consider the order of $l$ in $(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_{i=1}^{m} (\mathbb{Z}/M_i\mathbb{Z})^\times$, we see that the first condition is satisfied. For the second condition (if $a_0 > 0$), we need to make $M_0$ satisfy the following extra property:

$$M_0 \nmid l^{2^{t_0-1}} + 1.$$

Now we work on $i = 0$ first. Take $t_0 > a_0$ large enough such that $t_0 > 2$ and for each rational prime $q \in S \cup \{2\}$, one of the following holds:

(1) $q \nmid l^{2^k} - 1$ for any $k > 0$,

(2) $q \mid l^{2^{t_0 - 3}} - 1$.

The fact $t_0 > 2$ gives $l^{2^{t_0 - 2}} + 1 \equiv 2 \bmod 4$ and $l^{2^{t_0 - 1}} + 1 \equiv 2 \bmod 4$. Thus we may choose an odd prime divisor $A$ of $l^{2^{t_0 - 2}} + 1$ and an odd prime divisor $B$ of $l^{2^{t_0 - 1}} + 1$. We deduce that

$$AB \mid l^{2^{t_0}} - 1, \quad B \nmid l^{2^{t_0 - 1}} - 1, \quad A \nmid l^{2^{t_0 - 1}} + 1.$$

Take $M_0 = AB$. Thus the smallest $r$ such that $M_0 \mid l^r - 1$ is $2^{t_0}$ and $M_0 \nmid l^{2^{t_0 - 1}} + 1$. Also, for $q \in S$, if $q \mid M_0$, then $q \mid l^{2^{t_0 - 2}} + 1$ or $q \mid l^{2^{t_0 - 1}} + 1$. In either case (1) won't happen, so $q \mid l^{2^{t_0 - 3}} - 1$. Thus $q = 2$. And this gives a contradiction with $AB$ being odd. We have constructed an $M_0$ with the property stated above.

Now we inductively construct $M_i$ and $t_i \geq a_i$ such that

- $M_i$ is not divisible by any rational primes in $S_i = \{p_i\} \cup S \cup \{$rational prime divisors of $M_j$ for $j < i\} \cup \{l\} \cup \{2\}$.
- The order of $l$ in $(\mathbb{Z}/M_i\mathbb{Z})^\times$ is $p_i^{t_i}$.

Choose $t_i > a_i$ large enough, such that $l^{p_i^{t_i - 2}} > p_i$ and for each rational prime $q \in S_i$, one of the following holds:

(1) $q \nmid l^{p_i^k} - 1$ for any $k > 0$,

(2) $q \mid l^{p_i^{t_i - 2}} - 1$.

If $q \in S_i$ and $q \mid l^{p_i^{t_i - 1}(p_i - 1)} + \ldots + l^{p_i^{t_i - 1}} + 1$, then $q \mid l^{p_i^{t_i}} - 1$ and so (1) cannot hold. Hence $q \mid l^{p_i^{t_i - 2}} - 1$. Thus $l^{p_i^{t_i - 1}(p_i - 1)} + \ldots + l^{p_i^{t_i - 1}} + 1 \equiv p_i \bmod q$. We see $q = p_i$. In this case, $l^{p_i^{t_i - 1}} \equiv (l^{p_i^{t_i - 2}})^{p_i} \equiv (1 + p_i u)^{p_i} \equiv 1 \bmod p_i^2$ for some integer $u$, so $l^{p_i^{t_i - 1}(p_i - 1)} + \ldots + l^{p_i^{t_i - 1}} + 1 \equiv p_i \bmod p_i^2$ and $p_i^2 \nmid l^{p_i^{t_i - 1}(p_i - 1)} + \ldots + l^{p_i^{t_i - 1}} + 1$. Thus, the only prime in $S_i$ that divides $l^{p_i^{t_i - 1}(p_i - 1)} + \ldots + l^{p_i^{t_i - 1}} + 1$ is $p_i$ and only to the first order. We may take an odd prime divisor $M_i$ of $l^{p_i^{t_i - 1}(p_i - 1)} + \ldots + l^{p_i^{t_i - 1}} + 1 (> p_i)$ with $M_i \notin S_i$. Now $M_i \mid l^{p_i^{t_i}} - 1$, but if $M_i \mid l^{p_i^{t_i - 1}} - 1$, then $l^{p_i^{t_i - 1}(p_i - 1)} + \ldots + l^{p_i^{t_i - 1}} + 1 \equiv p_i \bmod M_i$. So $M_i = p_i$ giving a contradiction. The two condition on $M_i$ is thus satisfied.

Now take $N = \prod_{i=0}^m M_i$ as promised. The smallest positive integer $r$ such that $N \mid l^r - 1$ is $2^{t_0} \prod_{i=1}^m p_i^{t_i}$, a multiple of $s$. For the second condition, we

need to verify that $N \nmid l^{r/2} - 1 = l^{2^{t_0-1}} \prod_{i=1}^{m} p_i^{t_i} - 1$ (if $t_0 > 0$). We break into two cases: If $m > 0$, i.e. there are odd prime divisors of $s$, then from $M_1 \mid l^{2^{t_0-1}} \prod_{i=1}^{m} p_i^{t_i} - 1$ and $M_1$ is an odd divisor of $N$, we see $N \nmid l^{r/2} + 1$. If $m = 0$, then the construction stops at the first step and we have seen that $M_0 = N \nmid l^{2^{t_0-1}} + 1$.

$\square$

The following lemma is taken from [4] Lemma 2.2. The lemma will be used to prove that certain representations coming from the Dwork motive are ordinary.

**Lemma 2.4** *Suppose that $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l),+}$ and that*

$$r : \mathrm{Gal}(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}}_l)$$

*is crystalline at all primes $v \mid l$. We think of $v$ as a valuation $v : F_v^\times \twoheadrightarrow \mathbb{Z}$. If $\tau : F \to \overline{\mathbb{Q}}_l$ lies above $v$, suppose that*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i (r \otimes_{\tau, F_v} B_{dR})^{\mathrm{Gal}(\overline{F}_v/F_v)} = 0$$

*unless $i = a_{\tau,j} + n - j$ for some $j = 1, \ldots, n$, in which case*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i (r \otimes_{\tau, F_v} B_{dR})^{\mathrm{Gal}(\overline{F}_v/F_v)} = 1.$$

*For $v \mid l$, let $\alpha_{v,1}, \ldots, \alpha_{v,n}$ denote the roots of the characteristic polynomial of $\phi^{[F_v^0:\mathbb{Q}_l]}$ on*

$$(r \otimes_{\tau, F_v^0} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{F}_v/F_v)}$$

*for any $\tau : F_v^0 \hookrightarrow \overline{\mathbb{Q}}_l$. (Here $F_v^0$ is the maximal unramified subextension in $F_v$. This characteristic polynomial is independent of the choice of $\tau$.) Let $\mathrm{val}_v$ denote the valuation on $\overline{\mathbb{Q}}_l$ normalized by $\mathrm{val}_v(l) = v(l)$. (Thus $\mathrm{val}_v \circ \tau = v$ for any $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l$.) Arrange the $\alpha_{v,i}$'s such that*

$$\mathrm{val}_v(\alpha_{v,1}) \geq \mathrm{val}_v(\alpha_{v,2}) \geq \ldots \geq \mathrm{val}_v(\alpha_{v,n}).$$

*Then $r$ is ordinary of weight $a$ if and only if for all $v \mid l$ and all $i = 1, \ldots, n$ we have*

$$\mathrm{val}_v(\alpha_{v,i}) = \sum_{\tau} (a_{\tau,i} + n - i),$$

*where $\tau$ runs over embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$ above $v$.*

*Remark 2.5* We will use $D_{\mathrm{cris},\tau}(r)$, $D_{\mathrm{st},\tau}(r)$ to denote $(r \otimes_{\tau, F_v^0} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{F}_v/F_v)}$, $(r \otimes_{\tau, F_v^0} B_{\mathrm{st}})^{\mathrm{Gal}(\overline{F}_v/F_v)}$ resp. for any $p$-adic representation $r$ and embedding $\tau$ as above.

Below is a lemma regarding the standard big image condition and how they behave under restriction to smaller absolute Galois group and taking symmetric power. The lemma is taken from an old version of [1], but a part of it disappears in the newest version of [1]. Thus, we state and prove it in this section. The author claims no originality of the following lemma.

Throughout this paper, for any number field $M$, we use $\widetilde{M}$ to denote its normal closure over $\mathbb{Q}$.

**Lemma 2.6** *Suppose $F/\mathbb{Q}$ is a finite extension with normal closure $\widetilde{F}/\mathbb{Q}$ and $n \in \mathbb{Z}_{\geq 0}$. Suppose also that $l > 2n + 5$ is a rational prime and that $\overline{r} : G_F \to GL_2(\overline{\mathbb{F}}_l)$ is a continuous representation such that $\overline{r}(G_{\widetilde{F}}) \supset SL_2(\mathbb{F}_l)$. Finally assume $F_1/F$ is a finite extension that is linearly disjoint from $\overline{F}^{\ker \overline{r}}$ over $F$. Then:*

*(1) $(\mathrm{Symm}^{n-1}\overline{r})(G_{F_1(\zeta_l)})$ is enormous.*
*(2) If $F_1/F$ is Galois and linearly disjoint over $F$ from the normal closure $H'$ of $H = \widetilde{F}\overline{F}^{\ker \mathrm{ad}\overline{r}}$ over $\mathbb{Q}$, then $\mathrm{Symm}^{n-1}\overline{r}|_{G_{F_1}}$ is decomposed generic.*

*Proof* (1) This is Lemma 7.1.6(2) of [1].

(2) It suffices to show $\mathrm{Symm}^{n-1}\overline{r}_{G_{F_2}}$ is decomposed generic for some finite extension $F_2/F_1$.

We may assume without loss of generality that $F$ is Galois over $\mathbb{Q}$. Now by [6] Theorem 2.47(b), $\mathrm{ad}\overline{r}(G_F) = PGL_2(k)$ or $PSL_2(k)$ for some finite extension $k/\mathbb{F}_l$. We first take an at most qurdratic extension $E/F$ such that $\mathrm{ad}\overline{r}(G_E) = PSL_2(k)$. Then by Goursat Lemma, $\mathrm{Gal}(\widetilde{E}/E) = (\mathbb{Z}/2\mathbb{Z})^r$ for some $r \geq 0$. Hence $\overline{F}^{\ker \mathrm{ad}\overline{r}}$ and $\widetilde{E}$ are linearly disjoint over $E$ by an analysis of the simple factors of each Galois group. Thus $\mathrm{ad}\overline{r}(G_{\widetilde{E}}) = PSL_2(k) \supset PSL_2(\mathbb{F}_l)$. Now since $E \subset \overline{F}^{\ker \mathrm{ad}\overline{r}}$, $\widetilde{E}\widetilde{E}^{\ker \mathrm{ad}\overline{r}} \subset \widetilde{\overline{F}^{\ker \mathrm{ad}\overline{r}}} = H'$, and so its normal closure over $\mathbb{Q}$ is still $H'$. Now $F_1$ and $H'$ linearly disjoint over $F$ implies that $E_1 := \widetilde{E}F_1$ is linearly disjoint from $H'$ over $\widetilde{E}$ as well. We therefore can assume (replacing $F$ by $\widetilde{E}$ and $F_1$ by $E_1$) without loss of generality that $\mathrm{ad}\overline{r}(G_F) = PSL_2(k)$ and $F$ is Galois over $\mathbb{Q}$.

Now we choose a sequence of subfields $F = F_0' \subset F_1' \subset \cdots \subset F_s' = F_1$ such that $F_i'$ is Galois over $F_{i-1}'$ and $\mathrm{Gal}(F_i'/F_{i-1}')$ is simple. Set $\widetilde{F}_i'$ to be the normal closure of $F_i'$ over $\mathbb{Q}$. Hence $\mathrm{Gal}(\widetilde{F}_i'/\widetilde{F}_{i-1}')$ is trivial if and only if $F_i' \subset \widetilde{F}_{i-1}'$ and is of form $\Delta_i^m$ for some $m > 0$ where $\Delta_i = \mathrm{Gal}(F_i'/F_{i-1}')$ otherwise (Goursat Lemma). Now if $H \cap \widetilde{F}_s' = F$, then we may apply Lemma 7.1.6 (3) of [1] to conclude.

Otherwise, there should exist a minimal $i$ such that $H \cap \widetilde{F}'_i \neq F$. Since $\mathrm{Gal}(H/F) = PSL_2(k)$ is simple, we see that $H \subset \widetilde{F}'_i$. Now minimality gives $\widetilde{F}'_{i-1} \cap H = F$, so that $\Delta_i^m = \mathrm{Gal}(\widetilde{F}'_i/\widetilde{F}'_{i-1}) \twoheadrightarrow \mathrm{Gal}(H/F)$, thus $\Delta_i = PSL_2(k)$, $m > 0$. We claim there exists $\sigma \in G_{\mathbb{Q}}$ such that $\sigma H \subset F'_i \widetilde{F}'_{i-1}$. We may write $\widetilde{F}'_i$ as the composite of $\sigma_j F'_i \widetilde{F}'_{i-1}$, where $\sigma_1, \ldots, \sigma_m$ are elements of $G_{\mathbb{Q}}$, the fields $\sigma_j F'_i \widetilde{F}'_{i-1}$ is Galois over $\widetilde{F}'_{i-1}$ with Galois group $PSL_2(k)$, and for any two disjoint subsets $I, J$ of $\{1, \ldots, m\}$, the composite of $\sigma_j F'_i \widetilde{F}'_{i-1}$ for $j \in I$ and the composite of $\sigma_j F'_i \widetilde{F}'_{i-1}$ for $j \in J$ are linearly disjoint over $\widetilde{F}'_{i-1}$. Now because $H \subset \widetilde{F}'_i$ and $\widetilde{F}'_{i-1} \cap H = F$, we see that $H\widetilde{F}'_{i-1}$ is a Galois subextension of $\widetilde{F}'_i/\widetilde{F}'_{i-1}$ with Galois group $PSL_2(k)$. We may pick the smallest $j$ such that $E_j := (\sigma_1 F'_i) \cdots (\sigma_j F'_i)\widetilde{F}'_{i-1} \supset H\widetilde{F}'_{i-1}$. The minimality gives that $E_{j-1}$ as a Galois extension of $\widetilde{F}'_{i-1}$ does not contain $H\widetilde{F}'_{i-1}$, whose Galois group over $\widetilde{F}'_{i-1}$ is simple. It follows that $E_{j-1}$ is linearly disjoint with $H\widetilde{F}'_{i-1}$ over $\widetilde{F}'_{i-1}$. Therefore, restriction map takes $\mathrm{Gal}(E_j/E_{j-1})$ onto $\mathrm{Gal}(H\widetilde{F}'_{i-1}/\widetilde{F}'_{i-1})$. Observe that $E_{j-1}$ and $\sigma_j F'_i \widetilde{F}'_{i-1}$ are linearly disjoint over $\widetilde{F}'_{i-1}$, hence $\mathrm{Gal}(E_j/\widetilde{F}'_{i-1}) = \mathrm{Gal}(E_j/E_{j-1}) \times \mathrm{Gal}(E_j/\sigma_j F'_i \widetilde{F}'_{i-1})$. The latter group $\mathrm{Gal}(E_j/\sigma_j F'_i \widetilde{F}'_{i-1})$ commutes with $\mathrm{Gal}(E_j/E_{j-1})$ inside $\mathrm{Gal}(E_j/\widetilde{F}'_{i-1})$. Hence under restriction map, by the surjective result proved above, $\mathrm{Gal}(E_j/\sigma_j F'_i \widetilde{F}'_{i-1})$ maps into the center of $\mathrm{Gal}(H\widetilde{F}'_{i-1}/\widetilde{F}'_{i-1})$, which is trivial. This gives us that $\sigma_j F'_i \widetilde{F}'_{i-1} \supset H\widetilde{F}'_{i-1}$. Thus taking $\sigma = \sigma_j^{-1}$ is sufficient for our claim.

Consider the image of $\mathrm{Gal}(\widetilde{F}'_{i-1} F'_i/F'_i)$ in $\mathrm{Gal}(\sigma H/F)$ under the natural restriction map. The fact $m > 0$ gives that $F'_i$ and $\widetilde{F}'_{i-1}$ are linearly disjoint over $F'_{i-1}$, and so $\mathrm{Gal}(\widetilde{F}'_{i-1} F'_i/F'_i)$ and $\mathrm{Gal}(\widetilde{F}'_{i-1} F'_i/\widetilde{F}'_{i-1})$ are commuting subgroups of $\mathrm{Gal}(\widetilde{F}'_{i-1} F'_i/F'_{i-1})$. Under the restriction map to $\mathrm{Gal}(\sigma H/F)$, since $\sigma H \cap \widetilde{F}'_{i-1} = F$, $\mathrm{Gal}(\widetilde{F}'_{i-1} F'_i/\widetilde{F}'_{i-1})$ surjects onto $\mathrm{Gal}(\sigma H/F)$, so the image of $\mathrm{Gal}(\widetilde{F}'_{i-1} F'_i/F'_i)$ lies in the center of $\mathrm{Gal}(\sigma H/F) \cong PSL_2(k)$, and hence is trivial. Therefore, $\sigma H \subset F'_i$, which contradicts the condition that $H'$ and $F_1$ are linearly disjoint over $F$. $\qquad\square$

## 3 Dwork motives

In this section, $l$ can be any odd prime, $n$ be any integer $\geq 2$ and $N$ is an integer that is

- odd, not divisible by any prime factors of $ln$.
- $N > 100n + 100$

but note that the case $n > 2$ and $n = 2$ differs a little bit, in that there will be a slight change of the category where the objects we considered lie in. The

condition of $N$ is clearly not optimal. However, since $N$ will be an integer that comes from a choice as in Sect. 4, rather than being given, it would not affect the generality of our main results.

We assume in this section that $F$ is a CM number field containing $\zeta_N$.

We will modify the construction and argument in Sect. 4 of [4] to fit the situation where no self-duality holds.

Let $T_0 = \mathbb{P}^1 - (\{\infty\} \cup \mu_N)/\mathbb{Z}[1/N]$ with coordinate $t$ and $Y \subset \mathbb{P}^{N-1} \times T_0$ be a projective family defined by the following equation:

$$X_1^N + X_2^N + \cdots + X_N^N = Nt X_1 X_2 \cdots X_N,$$

$\pi : Y \to T_0$ is a smooth of relative dimension $N - 2$. We will write $Y_s$ for the fiber of this family at a point $s$. Let $H = \mu_N^N/\mu_N$ where the second $\mu_N$ embeds diagonally and

$$H_0 := \{(\xi_1, \ldots, \xi_N) \in \mu_N^N : \xi_1 \cdots \xi_N = 1\}/\mu_N \subset H.$$

Over $\mathbb{Z}[1/N, \zeta_N]$ there is an $H$ action on $Y$ by:

$$(\xi_1, \ldots, \xi_N)(X_1, \ldots, X_N, t) = (\xi_1 X_1, \ldots, \xi_N X_N, (\xi_1 \cdots \xi_N)^{-1} t).$$

Thus $H_0$ acts on every fibre $Y_s$, and $H$ acts on $Y_0$.

Fix $\chi$ a character $H_0 \to \mu_N$ of the form:

$$\chi ((\xi_1, \ldots, \xi_N)) = \prod_{i=1}^{N} \xi_i^{a_i}$$

where

$$(a_1, \ldots, a_N) = (1, 2, 4, 5, \ldots, (N - n + 2)/2,$$
$$(N + n - 2)/2, \ldots,$$
$$N - 5, N - 4, N - 3, 0, 0, \ldots, 0)$$

when $n > 2$ is odd,

$$(a_1, \ldots, a_N) = (1, 2, 3, 4, 5, 7, 8, \ldots,$$
$$(N - n + 3)/2,$$
$$(N + n - 3)/2, (N + n - 1)/2, \ldots, N - 4, 0, 0, \ldots, 0)$$

when $n > 2$ is even, and

$$(a_1, \ldots, a_N) = (1, 2, \ldots (N - 3)/2, 0, 0, 0, (N + 3)/2, \ldots, N - 1)$$

when $n = 2$.

Note that $3, N - 2, N - 1$ do not occur in $(a_1, \ldots, a_N)$ when $n > 2$ is odd, $6, N - 3, N - 2, N - 1$ do not occur in $(a_1, \ldots, a_N)$ when $n > 2$ is even, and there are $n + 1$ of 0s in $(a_1, \ldots, a_N)$ in these cases $(n > 2)$.

This character is well-defined because $\sum_{i=1}^{N} a_i \equiv 0 \mod N$.

Let $(b_1, \ldots, b_n)$ be mutually distinct residue classes in $\mathbb{Z}/N\mathbb{Z}$ such that $b_i + a_j \neq 0$ in $\mathbb{Z}/N\mathbb{Z}$ for any $j \in \{1, \ldots, N\}$. The $(b_1, \ldots, b_n)$ are uniquely determined since there are precisely $n$ different residue classes whose sum with $a_j$ is nonzero in $\mathbb{Z}/N\mathbb{Z}$ for any $j \in \{1, \ldots, N\}$. Hence we have the following expression:

$$\{b_1, \ldots, b_n\} = \begin{cases} 1, 2, (N - n + 4)/2, (N - n + 6)/2, \ldots, \\ \quad (N + n - 6)/2, (N + n - 4)/2, N - 3 & n > 2 \text{ odd} \\ 1, 2, 3, (N - n + 5)/2, (N - n + 7)/2, \ldots, \\ \quad (N + n - 7)/2, (N + n - 5)/2, N - 6 & n > 2 \text{ even} \\ (N - 1)/2, (N + 1)/2 & n = 2 \end{cases}$$

(when $n = 3, 4$, this is interpreted as $\{b_1, \ldots, b_n\} = \{1, 2, N-3\}, \{1, 2, 3, N-6\}$ respectively.)

We have the following combinatorial property for the set $\{b_1, \ldots, b_n\} \subset \mathbb{Z}/N\mathbb{Z}$.

**Lemma 3.1** *Let $n > 2$. Consider $\{b_1, \ldots, b_n\}$ as a subset of $\{0, 1, \ldots, N - 1\} \cong \mathbb{Z}/N\mathbb{Z}$. If for some $\alpha \in (\mathbb{Z}/N\mathbb{Z})^\times$, $\{\alpha b_1, \ldots, \alpha b_n\} = \{b_1, \ldots, b_n\}$ holds, then $\alpha = 1$.*

*Proof* If $n$ is even, then $1 \in \{b_1, \ldots, b_n\}$ and so $\alpha \in \{b_1, \ldots, b_n\}$. If $\alpha = 2$, then $2 \in \{b_1, \ldots, b_n\}$ and so $4 \in \{b_1, \ldots, b_n\}$. But by the assumption $N > 100n + 100$, $3 < 4 < (N - n + 5)/2$, so that $4 \notin \{b_1, \ldots, b_n\}$. Thus $\alpha \neq 2$. Same argument $(3 < 9, 36 < (N - n + 5)/2)$ shows that $\alpha \neq 3, N - 6$. If $n = 4$, we are done. For $n \geq 6$, if $\alpha = (N + x)/2$ for some $x$ odd and $x \in [-n+5, n-5]$, then $2\alpha = x \in \{b_1, \ldots, b_n\}$, but by assumption on $N$ and $n$, we have $(N+n-5)/2 < N-n+5 \leq N-1$ and $1 \leq n-5 < (N-n+5)/2$. Therefore $\{1, 3, \ldots, n-5, N-n+5, N-n+7, \ldots, N-1\} \cap \{b_1, \ldots, b_n\} \subset \{1, 3\}$ and so $x = 1$ or $3$. In either case, if $\alpha = (N+x)/2 \in \{b_1, \ldots, b_n\}$, then $(N-x)/2 \in \{b_1, \ldots, b_n\}$, so $-x^2/4 \equiv (N+x)/2 \cdot (N-x)/2 \in \{b_1, \ldots, b_n\}$, thus $N - 1 \in \{4b_1, \ldots, 4b_n\}$ or $N - 9 \in \{4b_1, \ldots, 4b_n\}$. Viewed as a subset of the representatives $\{0, 1, \ldots, N - 1\}$, $\{4b_1, \ldots, 4b_n\} \subset \{2, 6, \ldots, 2n - 10, N - (2n - 10), N - (2n - 14), \ldots, N - 2\} \cup \{4, 8, 12, N - 24\}$. But $N - 1, N - 9 > 2n - 10, 12$, so they must lies in $\{N - (2n - 10), N - (2n - 14), \ldots, N - 2\} \cup \{N - 24\}$, a contradiction.

If $n$ is odd, again $1 \in \{b_1, \ldots, b_n\}$ and so $\alpha \in \{b_1, \ldots, b_n\}$. If $\alpha = 2$, then again $4 \in \{b_1, \ldots, b_n\}$. But $2 < 4 < (N - n + 4)/2$ gives a contradiction.

Similarly $\alpha \neq N - 3$ because $2 < 9 < (N - n + 4)/2$. If $n = 3$ we are done. For $n \geq 5$, we have $\alpha = (N + x)/2$ for some $x$ odd and $x \in [-n + 4, n - 4]$. Then $2\alpha = x \in \{b_1, \ldots, b_n\}$, but by assumption on $N$, we have $(N + n - 4)/2 < N - n + 4 \leq N - 1$ and $1 \leq n - 4 < (N - n + 4)/2$. Therefore $\{1, 3, \ldots, n-4, N-n+4, N-n+6, \ldots, N-1\} \cap \{b_1, \ldots, b_n\} \subset \{1, N-3\}$ and so $x = 1$ or $-3$. In either case, if $\alpha = (N + x)/2 \in \{b_1, \ldots, b_n\}$, then $(N-x)/2 \in \{b_1, \ldots, b_n\}$, so $-x^2/4 \equiv (N+x)/2 \cdot (N-x)/2 \in \{b_1, \ldots, b_n\}$, thus $N-1 \in \{4b_1, \ldots, 4b_n\}$ or $N-9 \in \{4b_1, \ldots, 4b_n\}$. Viewed as a subset of the representatives $\{0, 1, \ldots, N-1\}$, $\{4b_1, \ldots, 4b_n\} \subset \{2, 6, \ldots, 2n-8, N-(2n-8), N-(2n-12), \ldots, N-2\} \cup \{4, 8, N-12\}$. But $N-1, N-9 > 2n-8, 8$, so they must lie in $\{N-(2n-8), N-(2n-12), \ldots, N-2\} \cup \{N-12\}$, a contradiction. $\qquad\square$

One reason for this choice is that when $n > 2$ we want to avoid making the set $\{a_1, \ldots, a_N\}$ self-dual so that Lemmas 3.3 and 3.7 hold. And the above lemma will be crucial to avoid self duality. On the contrary, if the set $\{a_1, \ldots, a_N\}$ is chosen to be self-dual, the $V[\lambda]$ defined below would be a Galois representation that takes values in $GSp_n$.

For any prime $\lambda$ of $\mathbb{Z}[1/2N, \zeta_N]$ of residue characteristic $l$, we define the lisse sheaf $V_\lambda/(T_0 \times \operatorname{Spec}\mathbb{Z}[1/2Nl, \zeta_N])_{et}$ by:

$$V_\lambda = (R^{N-2}\pi_*\mathbb{Z}[\zeta_N]_\lambda)^{\chi, H_0}$$

when $n > 2$. When $n = 2$, we use the same formula to define the object $U_\lambda$ for a prime $\lambda$ of $\mathbb{Z}[1/2N, \zeta_N]^+$, following the notation of Sect. 4 in [4].

Similarly, for any nonzero ideal $\mathfrak{n}$ of $\mathbb{Z}[1/2N, \zeta_N]$ of norm $M$, we can define the lisse sheaf $V[\mathfrak{n}]/(T_0 \times \operatorname{Spec}\mathbb{Z}[1/2NM, \zeta_N])_{et}$ by:

$$V[\mathfrak{n}] = (R^{N-2}\pi_*(\mathbb{Z}[\zeta_N]/\mathfrak{n}))^{\chi, H_0}$$

when $n > 2$, and we use the same formula to define the object $U[\mathfrak{n}]$ for a nonzero ideal $\mathfrak{n}$ of $\mathbb{Z}[1/2N, \zeta_N]^+$.

Since $H$ acts on $Y_0$, we have the following decomposition:

$$V_{\lambda,0} = \bigoplus_{i=1}^{N} V_{\lambda,i}, \quad V[\mathfrak{n}]_0 = \bigoplus_{i=1}^{N} V_i[\mathfrak{n}]$$

here $V_{\lambda,i}$ and $V_i[\mathfrak{n}]$ are the subspace of $V_{\lambda,0}$, $V[\mathfrak{n}]$ where $H$ acts by the character $\chi_i$:

$$\xi \to \prod_{j=1}^{N} \xi_j^{a_j+i}$$

Again, we write same decomposition in the case $n = 2$ as into $U_{\lambda,i}$ and $U_i[\mathfrak{n}]$.

Fix an embedding $\tau : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$ such that $\tau(\zeta_N) = e^{2\pi i/N}$. Let $\tilde{\pi} : Y(\mathbb{C}) \to T_0(\mathbb{C})$ denote the base change of $\pi$ along $\tau$, viewed as a map of complex analytic spaces and $V_B$ be the locally constant sheaf over $T_0(\mathbb{C})$:

$$V_B = (R^{N-2}\tilde{\pi}_*\mathbb{Z}[\zeta_N])^{\chi, H_0}$$

when $n > 2$. And we denote the same object in the case $n = 2$ as $U_B$.

Let $\tau$ also denote the induced base change $(T_0)_\mathbb{C} \to T_0 \times \operatorname{Spec} \mathbb{Z}[1/2NMl, \zeta_N]$. Under previous notation, $V_B \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}[\zeta_N]_\lambda$ corresponds to $\tau^*V_\lambda$ under the equivalence between locally constant analytic $\mathbb{Z}[\zeta_N]_\lambda$-sheaves on $T_0(\mathbb{C})$ and locally constant etale $\mathbb{Z}[\zeta_N]_\lambda$-sheaves on $(T_0)_\mathbb{C}$.

Similarly, $V_B \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}[\zeta_N]/\mathfrak{n}$ corresponds to $\tau^*V[\mathfrak{n}]$ under the equivalence between locally constant analytic $\mathbb{Z}[\zeta_N]/\mathfrak{n}$-sheaves on $T_0(\mathbb{C})$ and locally constant etale $\mathbb{Z}[\zeta_N]/\mathfrak{n}$-sheaves on $(T_0)_\mathbb{C}$.

Similar relation holds when $n = 2$, see [4] Sect. 4.

Let $T_0' = \mathbb{P}^1 - \{0, 1, \infty\}$ with coordinate $t'$ and $Y' \subset \mathbb{P}^{N-1} \times T_0'$ be a projective family defined by the following equation:

$$X_1'^N + X_2'^N + \cdots + t'^{-1}X_N'^N = NX_1'X_2'\cdots X_N'$$

Then $t' \mapsto t^N$ gives an $N$-fold Galois covering $T_0 - \{0\} \to T_0'$ and $X_1' \to X_1, X_2' \to X_2 \ldots, X_N' \to tX_N$ identifies the pullback of $\pi' : Y' \to T_0'$ along this covering with $\pi : Y - Y_0 \to T_0 - \{0\}$. Over $\mathbb{Z}[1/N, \zeta_N]$, $H_0$ acts on $Y'$ by

$$(\xi_1, \ldots, \xi_N)(X_1', \ldots, X_N', t') = (\xi_1 X_1', \ldots, \xi_N X_N', t')$$

This $H_0$ action is compatible with the $H_0$ action on $Y - Y_0$.

Let $\tilde{\pi}' : Y'(\mathbb{C}) \to T_0'(\mathbb{C})$ be the base change of $\pi'$ along $\tau$ viewed as a map of complex analytic spaces and let $V_B' = (R^{N-2}\tilde{\pi}_*'\mathbb{Z}[\zeta_N])^{\chi, H_0}$ be a locally constant sheaf over $T_0'(\mathbb{C})$. Then the pullback of $V_B'$ along the covering $T_0(\mathbb{C}) - \{0\} \to T_0'(\mathbb{C})$ is naturally identified with $V_B$ over $T_0(\mathbb{C}) - \{0\}$.

Fix a nonzero base point $t \in T_0(\mathbb{C})$ and let $t'$ be its image in $T_0'(\mathbb{C})$. Now we study the image of the monodromy representation:

$$\rho_t : \pi_1(T_0(\mathbb{C}), t) \to GL(V_{B,t}).$$

We in turn consider the monodromy representation:

$$\rho_{t'} : \pi_1(T_0'(\mathbb{C}), t') \to GL(V_{B,t'}')$$

**Proposition 3.2** *The sheaves $V_\lambda$, $V[\mathfrak{n}]$, $V_B$, $V'_B$ are locally free over $\mathbb{Z}[\zeta_N]_\lambda$, $\mathbb{Z}[\zeta_N]/\mathfrak{n}$, $\mathbb{Z}[\zeta_N]$, $\mathbb{Z}[\zeta_N]$ of rank $n$ respectively .*

*Proof* Locally freeness follows because the family is smooth and proper. For the rank part, one only need to check at the fibre over 0 and apply Proposition 7.4 of [7].

$\square$

Similar relation holds in the case $n = 2$, see [4] Sect. 4. The different aspect for the case $n = 2$ is that we may use the locally constant sheaves $V_\lambda$, $V[\mathfrak{n}]$, $V_B$ defined there with coefficients $\mathbb{Z}[\zeta_N]^+_\lambda$, $\mathbb{Z}[\zeta_N]^+/\mathfrak{n}$, $\mathbb{Z}[\zeta_N]^+$ respectively, such that $V_\lambda \otimes_{\mathbb{Z}[\zeta_N]^+} \mathbb{Z}[\zeta_N]$, $V[\mathfrak{n}] \otimes_{\mathbb{Z}[\zeta_N]^+} \mathbb{Z}[\zeta_N]$, $V_B \otimes_{\mathbb{Z}[\zeta_N]^+} \mathbb{Z}[\zeta_N]$ are isomorphic to $U_\lambda$, $U[\mathfrak{n}]$, $U_B$ respectively (in their notation). Now we consistently work with $V_\lambda$, $V[\mathfrak{n}]$, $V_B$, regardless of whether $n > 2$ or $n = 2$.

We already have the counterpart of Lemma 3.7 as provided by Corollary 4.7 of [4]. i.e. $\overline{\rho}_t(\pi_1(T_0(\mathbb{C}), t)) = SL(V_{B,t}/\lambda)$ when $n = 2$. So we focus on the case $n > 2$ until the end of the proof of Lemma 3.7.

Let $\gamma_0$, $\gamma_1$, $\gamma_\infty$ be the loop around $0$, $1$, $\infty$, generating $\pi_1(T'_0(\mathbb{C}), t')$ subject only to the relation $\gamma_0\gamma_1\gamma_\infty = 1$. Here we let $\gamma_0$ be such oriented that its image in $\mathrm{Gal}(T_0(\mathbb{C}) - \{0\}/T'_0(\mathbb{C}))$ is $e^{2\pi i/N} = \tau(\zeta_N)$

**Lemma 3.3** *(1) $\rho_{t'}(\gamma_0)$ has characteristic polynomial $\prod_{j=1}^n (X - \zeta_N^{b_j})$ where
(2) $\rho_{t'}(\gamma_\infty)$ has characteristic polynomial $(X - 1)^n$.
(3) $\rho_{t'}(\gamma_1)$ is a transvection, i.e .: it is unipotent and $\ker(\rho_{t'}(\gamma_1) - 1)$ has dimension $n - 1$.*

*Proof* (1) The action of $\gamma_0$ on $V'_{B,t'}$, is equivalent to the $\zeta_N$ action on $V_{B,0}$, which is the scalar multiplication by $\zeta_N^i$ on the $\chi_i$ eigenspace of $V_{B,0}$. By proposition 7.4 of [7], the $\chi_i$-eigenspaces are nonzero if and only if $0 \notin \{i + a_1, \ldots, i + a_N\}$, i.e. $i \in \{b_1, \ldots, b_n\}$, in which case the eigenspaces are all of rank 1. Hence the expression of the characteristic polynomial of $\rho_{t'}(\gamma_0)$ follows.

(2) Suppose $Z_0$ is the variety $T(X_1^N + X_2^N + \cdots + X_N^N) = NX_1X_2\cdots X_N$ contained in $\mathbb{P}^{N-1} \times \mathbb{A}^1$. We use $p$ to denote the projection $Z_0 \to \mathbb{A}^1$. So it suffice to show the monodromy around 0 of the larger local system $R^{N-2}p_*\mathbb{C}$ has charatheristic polynomial a power of $(X - 1)$. We apply Lemma 2.1 of [14] base changed to $\mathbb{C}$ via $W(k)[T, U^\pm] \to \mathbb{C}[T]$, $U \mapsto 1$, $T \mapsto T$ and a fixed isomorphism of $\overline{W(k)[\frac{1}{p}]} \cong \mathbb{C}$, to conclude that there is a blowup $\mathbb{X}$ of $Z_0$ that is an isomorphism outside the fiber over 0 and $\mu_N$, and $\mathbb{X}$ is semistable over the base $\mathbb{A}^1\backslash\mu_N$. Note that we call a map semistable if the divisor $(T)$ is a reduced normal crossing divisor and does not have self crossing throughout this paper.

Because the blowup is an isomorphism outside the fiber over 0 and $\mu_N$, we are reduced to show the same monodromy result for the family $\mathbb{X}$ over $\mathbb{A}^1$ around 0. Thus, by the vanishing cycle technique used to prove local monodromy theorem (cf. [11] Theoreme 2.1.2 and its proof), we see that such monodromy is unipotent. Note that this makes use of the fact that our normal crossing divisor $(T)$ is reduced, i.e. the exponents $e_i$ as in the notation of [11] are all 1 and hence so is their greatest common divisor $e$.

(3) The proof is the same as part 2 of Lemma 4.3 in [4]. $\qquad\square$

Now we study the image of the monodromy map. Let $\lambda$ be a prime of $\mathbb{Z}[\zeta_N]$ (of $\mathbb{Z}[\zeta_N]^+$ when $n = 2$) of characteristic $l$ and $\overline{\rho}_{t'} : \pi_1(T'_0(\mathbb{C}), t') \to GL(V_{B,t'}/\lambda)$, $\overline{\rho}_t : \pi_1(T_0(\mathbb{C}), t) \to GL(V_{B,t}/\lambda)$ be the reduction of $\rho_{t'}$, $\rho_t$ by $\lambda$ respectively.

We first give a description of $\rho_{t'}$ by Lemma 3.3 and the following lemma.

**Lemma 3.4** *Let $\rho$ be the representation $\rho : \pi_1(T'_0(\mathbb{C}), t') \to GL_n(\mathbb{Z}[\zeta_N])$ sending $\gamma_0$ to $B^{-1}$, $\gamma_\infty$ to $A$, and $\gamma_1$ to $BA^{-1}$, where*

$$
A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_n \\ 1 & 0 & \cdots & 0 & -A_{n-1} \\ 0 & 1 & \cdots & 0 & -A_{n-2} \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix},
$$

$$
B = \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_n \\ 1 & 0 & \cdots & 0 & -B_{n-1} \\ 0 & 1 & \cdots & 0 & -B_{n-2} \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & -B_1 \end{pmatrix},
$$

*and $A_i$, $B_i \in \mathbb{Z}[\zeta_N]$ are the coefficients of the expansions:*

$$
(X - 1)^n = X^n + A_1 X^{n-1} + \cdots + A_n,
$$
$$
\prod_{i=1}^{n}(X - \zeta_N^{-b_i}) = X^n + B_1 X^{n-1} + \cdots + B_n.
$$

*Then as representation into $GL_n(\mathbb{C})$, $\rho_{t'}$ and $\rho$ are equivalent.*

*Proof* See Theorem 3.5 of [3]. Observe also that $\rho(\gamma_1) = BA^{-1}$ has the form

$$\begin{pmatrix} C_n & 0 \cdots 0\ 0 \\ C_{n-1} & 1 \cdots 0\ 0 \\ & \ddots \\ C_2 & 0 \cdots 1\ 0 \\ C_1 & 0 \cdots 0\ 1 \end{pmatrix}$$

with all the $C_i \in \mathbb{Z}[\zeta_N]$. $\qquad\qquad\square$

Note that the matrix $A$ actually has minimal polynomial $(X - 1)^n$ and is conjugate to $\rho_{t'}(\gamma_\infty)$. We have the following corollary.

**Corollary 3.5** $\rho_{t'}(\gamma_\infty)$ *has minimal polynomial* $(X - 1)^n$ *and hence so is the image of the monodromy around* $\infty$ *under* $\rho_t$.

In the following, we will use the terms "maximally unipotent" and "maximally nilpotent" to refer to the properties mentioned above.

**Definition 3.6** For a unipotent (resp. nilpotent) linear operator $\phi$ on an $n$-dimensional vector space, we say it is maximally unipotent (resp. maximally nilpotent) if its minimal polymial is $(X - 1)^n$ (resp. $X^n$).

Let $\overline{\rho} : \pi_1(T'_0(\mathbb{C}),\ t') \to GL_n(\mathbb{Z}[\zeta_N]/\lambda)$ be the reduction of $\rho$ with respect to $\lambda$. (Following the argument of proposition 3.3 of [3]) Then if $\overline{\rho}$ has block upper-triangular form when base changed to $\overline{k}(\lambda)$, we see $\overline{\rho}(\gamma_1) - 1$ would vanish on one of the two blocks since it is a transvection, so that the eigenvalue of $\overline{\rho}(\gamma_0)$ and $\overline{\rho}(\gamma_\infty)$ would be the same on that block, which gives a contradiction because none of the $b_i$ is 0. Thus $\overline{\rho}$ is absolutely irreducible. Let $\overline{\rho}_{t'} : \pi_1(T'_0(\mathbb{C}),\ t') \to GL(V_{B,t},\ /\lambda)$ be the reduction of $\rho_{t'}$ by $\lambda$. It has the same trace with $\overline{\rho}$ by Lemma 3.4. So their semisimplification are equivalent and thus they are equivalent and $\overline{\rho}_{t'}$ is absolutely irreducible.

**Lemma 3.7** *Assume the residue field* $k(\lambda)$ *of* $\lambda$ *is* $\mathbb{F}_{l^r}$ *(So* $r$ *is the smallest integer such that* $N \mid l^r - 1$*). Under the assumption that* $N \nmid l^{r/2} + 1$ *if* $r$ *is even and* $n > 2$*, we have that* $\overline{\rho}_{t'}(\pi_1(T'_0(\mathbb{C}), t')) = SL(V'_{B,t},\ /\lambda)$ *and* $\overline{\rho}_t(\pi_1(T_0(\mathbb{C}), t)) = SL(V_{B,t}/\lambda)$.

*Proof* The case when $n = 2$ is already resolved by Lemma 4.6 of [4]. We now focus on the case $n > 2$.

Let $H$ be the normal subgroup of $\pi_1(T'_0(\mathbb{C}), t')$ generated by $\gamma_1$. Then $\pi_1(T'_0(\mathbb{C}),\ t')/H$ is cyclic, and is generated by $\gamma_0 H$ or $\gamma_\infty H$. Therefore the index $[\overline{\rho}_{t'}(\pi_1(T'_0(\mathbb{C}),\ t')) : \overline{\rho}_{t'}(H)]$ divides both the order of $\overline{\rho}_{t'}(\gamma_0)$ and $\overline{\rho}_{t'}(\gamma_\infty)$. The former is a divisor of $N$ and the latter is an $l$-power, thus

$\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t')) = \overline{\rho}_{t'}(H)$. So $\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t'))$ is generated by transvections, hence by the main theorem of [18], $\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t'))$ is conjugate in $GL_n(k(\lambda))$ to one of the groups $SL_n(k)$, $Sp_n(k)$ or $SU(n, k)$ for some subfield $k \subset k(\lambda)$. Here $SU(n, k)$ is defined when $[k : \mathbb{F}_l]$ even:

$$SU(n, k) := \{g \in SL_n(k) : \sigma(g)^t g = 1_n\},$$

where $\sigma$ is the unique order 2 element in $\mathrm{Gal}(k/\mathbb{F}_l)$. We want to show $\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t')) = SL_n(k(\lambda))$ by excluding the other cases.

- If $\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t'))$ is conjugate in $GL_n(k(\lambda))$ to one of the groups $SL_n(k)$, $Sp_n(k)$ or $SU(n, k)$ for some proper subfield $k \subsetneq k(\lambda)$, then there exists a nontrivial $\sigma \in \mathrm{Gal}(k(\lambda)/\mathbb{F}_l)$ that preserve the eigenvalues for any elements in $\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t'))$. Consider $\overline{\rho}_{t'}(\gamma_0)$, this would contradict Lemma 3.1.
- If $\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t'))$ is conjugate to $Sp_n(k(\lambda))$, by Proposition 6.1 of [3], we have

$$\{b_1, \ldots, b_n\} = \{-b_1, \ldots, -b_n\},$$

which contradicts Lemma 3.1.
- If $\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t'))$ is conjugate to $SU(n, k(\lambda))$, then we are in the situation $r = [k(\lambda) : \mathbb{F}_l]$ is even. Take the eigenvalue of both sides of the equation $\sigma(\overline{\rho}_{t'}(\gamma_0)) = ({}^t\overline{\rho}_{t'}(\gamma_0))^{-1}$. we have

$$\{l^{r/2}b_1, \ldots, l^{r/2}b_n\} = \{-b_1, \ldots, -b_n\}$$

By Lemma 3.1, we must have $l^{r/2} \equiv -1 \bmod N$. This contradicts the condition that $N \nmid l^{r/2} + 1$ if $r$ is even.

Thus $\overline{\rho}_{t'}(\pi_1(T_0'(\mathbb{C}), t')) = SL_n(k(\lambda))$. View $\overline{\rho}_t$ as defined on $\pi_1(T_0(\mathbb{C}) - \{0\}, t)$ via the surjection $\pi_1(T_0(\mathbb{C}) - \{0\}, t) \to \pi_1(T_0(\mathbb{C}), t)$. Since $\pi_1(T_0(\mathbb{C}) - \{0\}, t) \lhd \pi_1(T_0'(\mathbb{C}), t')$ with quotient group cyclic of order $N$, we have $\overline{\rho}_t(\pi_1(T_0(\mathbb{C}) - \{0\}, t)) \lhd SL_n(k(\lambda))$ with quotient cyclic of order dividing $N$. Now as the only cyclic composition factor of $SL_n(k(\lambda))$ have order dividing $n$, we see $\overline{\rho}_t(\pi_1(T_0(\mathbb{C}), t)) = \overline{\rho}_t(\pi_1(T_0(\mathbb{C}) - \{0\}, t)) = SL_n(k(\lambda))$. $\square$

Given any nonzero ideal $\mathfrak{n}$ of $\mathbb{Z}[1/2N, \zeta_N]$ (of $\mathbb{Z}[1/2N, \zeta_N]^+$ when $n = 2$) and any finite free rank-$n$ $\mathbb{Z}[\zeta_N]/\mathfrak{n}$ ($\mathbb{Z}[\zeta_N]^+/\mathfrak{n}$ when $n = 2$)-module $W$ with a continuous $G_F$-action, we can view $W$ as a lisse sheaf on $(\mathrm{Spec}\ F)_{et}$. Now $\wedge^n V[\mathfrak{n}]$ is a lisse sheaf over $(T_0)_F$ of rank 1, and the associated monodromy representation $\det \rho : \pi_1(T_0, t) \to GL(\wedge^n V[\mathfrak{n}]_t)$ restricted to $\pi_1^{\mathrm{geom}}(T_0, t)$ is trivial since $\det(\gamma_0) = \det(\gamma_1) = \det(\gamma_\infty) = 1$ and the analytic $\pi_1$ is dense in $\pi_1^{\mathrm{geom}}(T_0, t)$. Thus $\det \rho$ factors through $\pi_1(\mathrm{Spec}\ F) = G_F$.

Suppose we are given an isomorphism of lisse sheaf over $(T_0)_F$ via some prescribed isomorphism of $G_F$ characters:

$$\phi : \bigwedge^n W_{(T_0)_F} \to \bigwedge^n V[\mathfrak{n}].$$

Let $\phi_S$ denote the base change of $\phi$ to some scheme $S$ over $(T_0)_F$. Define the moduli functor $T_W$ as following:

$$T_W : (\text{Sch}/(T_0)_F) \to (\text{Sets})$$
$$S \mapsto \{\psi \in \text{Isom}_S(W_S, V[\mathfrak{n}]_S) : \wedge^n \psi = \phi_S\}$$

It is representable by a smooth $T_W/(T_0)_F$.

**Proposition 3.8** *Under the notation and assumption above, if $\mathfrak{n} = \mathfrak{P}_1\mathfrak{P}_2$, where $\mathfrak{P}_1, \mathfrak{P}_2$ are two prime ideals of $\mathbb{Z}[\zeta_N]$ having different residue characteristic $l_1, l_2$ (prime to $2N$) respectively. If each of the $l_i$ satisfy the following condition:*

- *if $n > 2$ the smallest positive $r$ such that $N \mid l_i^r - 1$ is even, then $N \nmid l_i^{r/2} + 1$.*

*and $\max\{l_1, l_2\} > 10$, then $T_W$ is geometrically connected.*

*Proof* Since $\pi_1(T_0(\mathbb{C}), t) \to SL(V_{B,t}/\mathfrak{P}_1)$ and $\pi_1(T_0(\mathbb{C}), t) \to SL(V_{B,t}/\mathfrak{P}_2)$ are surjective by Lemma 3.7 and our condition, by Goursat Lemma we see that there exist isomorphic quotient $\phi : SL_n(\mathbb{F}_{l_1^r})/H_1 \cong SL_n(\mathbb{F}_{l_2^s})/H_2$ such that the image of $\pi_1(T_0(\mathbb{C}), t)$ in $SL(V_{B,t}/\mathfrak{n})$ is the preimage of the diagonal $\{(t, \phi(t)) \in SL_n(\mathbb{F}_{l_1^r})/H_1 \times SL_n(\mathbb{F}_{l_2^s})/H_2\}$ under the natural quotient map. Here we let the residue field of $\mathfrak{P}_1, \mathfrak{P}_2$ be $\mathbb{F}_{l_1^r}, \mathbb{F}_{l_2^s}$ respectively.

Assume without loss of generality that $l_1 > 10$. Then the only proper normal subgroups of $SL_n(\mathbb{F}_{l_1^r})$ are contained in its center and the quotient group $PSL_n(\mathbb{F}_{l_1^r})$ is a simple group. Thus if $SL_n(\mathbb{F}_{l_1^r})/H_1$ is not trivial, then it must have a Jordan-Holder factor isomorphic to $PSL_n(\mathbb{F}_{l_1^r})$. Since $l_1 > 10$, any Jordan-Holder factor of $SL_n(\mathbb{F}_{l_2^s})$ with $l_2 \neq l_1$ is not isomorphic to $PSL_n(\mathbb{F}_{l_1^r})$ since $PSL_n(\mathbb{F}_{l_1^r})$ is simple non-Abelian and $PSL_n(\mathbb{F}_{l_1^r})$ is not isomorphic to $PSL_n(\mathbb{F}_{l_2^s})$ by Theorem 2 of [2]. This contradiction gives us $SL_n(\mathbb{F}_{l_1^r})/H_1 = 1$ and the map $\pi_1(T_0(\mathbb{C}), t) \to SL(V_{B,t}/\mathfrak{n})$ is surjective.

Hence for any $t \in T_0(\mathbb{C})$ and any two geomereic points of $T_W$ above it which correspond to two isomorphisms $\psi_1, \psi_2 : W \to V[\mathfrak{n}]_t$ that respects $\phi$ (not necessarily respecting any Galois action because the points are geometric, hence such points always exist), we can pick a path $\gamma \in \pi_1^{\text{geom}}(T_0, t)$ such that its image under the monodromy map is $\psi_2 \circ \psi_1^{-1}$. Going along $\gamma$ induces a path in $T_W(\mathbb{C})$ connecting $\psi_1$ and $\psi_2$ (viewed as points in $T_W(\mathbb{C})$), so geometrically connectivity follows. $\square$

**Lemma 3.9** *Let $\zeta_N \in F$, $\lambda$ is a prime of $\mathbb{Q}(\zeta_N)$ ($\mathbb{Q}(\zeta_N)^+$ when $n = 2$) over the fixed prime $l$ and $k(\lambda)$ be the residue field, then viewing $\det V[\lambda]$ as a $G_F$ representation as explained above, we have $\det V[\lambda](G_F) \subset (\mathbb{F}_{l^2}^{\times} k(\lambda)^{\times})^n$.*

*Proof* By the analysis before, it suffices to calculate the $G_F$ action on $V_{\lambda,0} = \oplus_{i=1}^{N} V_{\lambda,i} = \oplus_{i=1}^{N} H^{N-2}(Y_0, \mathbb{Z}[\zeta_N]_\lambda)^{\chi_i, H}$, where $Y_0$ is the Fermat hypersurface $X_1^N + \cdots + X_N^N = 0$ in $\mathbb{P}^{N-1}$ and $\chi_i : H \to \mu_N$ is a character defined by:

$$\xi = (\xi_1, \ldots, \xi_N) \mapsto \prod_{j=1}^{N} \xi_j^{a_j + i}.$$

By [7] Proposition 7.10, $V_{\lambda,i} \neq 0$ (in fact 1-dimensional) only when $i \in \{b_1, \ldots, b_n\}$, and $\mathrm{Frob}_v$ acts on it by a scalar $q^{-1} \prod_{j=1}^{N} g(v, a_j + i)$, where $v$ is any place of $F$ whose residue characteristic does not divide $N$ or $l$, $q = \#k(v)$, and $g(v, a)$ (for $a \in \mathbb{Z}/N\mathbb{Z}$) is the Gauss sum defined with respect to an fixed additive character $\psi : \mathbb{F}_q \to (\overline{\mathbb{Q}}_l)^{\times}$:

$$g(v, a) = - \sum_{x \in \mathbb{F}_q^{\times}} t(x^{\frac{1-q}{N}})^a \psi(x)$$

here we also fix an isomorphism $t$ from the group of $N$-th roots of unity in $\mathbb{F}_q^{\times}$ and the group of $N$-th roots of unity in $\overline{\mathbb{Q}}_l$. We remark that each $g(v, a)$ depends on the choice of $\psi$ but $q^{-1} \prod_{j=1}^{N} g(v, a_j + i)$ does not.

Thus $\mathrm{Frob}_v$ acts as

$$q^{-n} \prod_{j=1}^{n} \prod_{i=1}^{N} g(v, a_i + b_j)$$

under $\det V_{\lambda,0}$.

Considering the choice of $a_i$ and $b_j$, the product can be rewritten as

$$\prod_{j=1}^{n} \prod_{i=1}^{N} g(v, a_i + b_j) = (\prod_{j=1}^{n} g(v, b_j))^n \prod_{j=1}^{n} \prod_{s \neq -b_k, \forall k} g(v, s + b_j)$$

$$= (\prod_{j=1}^{n} g(v, b_j))^n (\prod_{s \neq 0} g(v, s))^n /$$

$$\prod_{i,j\in\{1,...,n\},i\neq j} g(v, b_j - b_i)$$

$$= (\prod_{j=1}^{n} g(v, b_j))^n (\prod_{s\neq 0} g(v, s))^n / \prod_{i,j\in\{1,...,n\},i<j} q$$

$$= (\prod_{j=1}^{n} g(v, b_j))^n (q^{(N-1)/2})^n / q^{n(n-1)/2} \qquad (3.1)$$

for $v$ whose residue characterisitc is odd, and here $s$ always ranging through the residue class in $\mathbb{Z}/N\mathbb{Z}$. In the last two steps, we use that for any nonzero $a \in \mathbb{Z}/N\mathbb{Z}$,

$$g(v, a)g(v, -a) = (-1)^{a\frac{1-q}{N}} q = q.$$

We further verify that $\prod_{j=1}^{n} g(v, b_j) \in \mathbb{Q}_l(\zeta_N)$ by checking: $\forall \sigma \in G_{\mathbb{Q}_l(\zeta_N)}$, if $\sigma(\zeta_p) = \zeta_p^a$ ($p$ is the residue characteristic of $v$), then

$$\sigma(\prod_{j=1}^{n} g(v, b_j)) = \sigma(\prod_{j=1}^{n} \sum_{x\in\mathbb{F}_q^\times} -t(x^{\frac{1-q}{N}})^{b_j} \psi(x))$$

$$= \prod_{j=1}^{n} \sum_{x\in\mathbb{F}_q^\times} -t(x^{\frac{1-q}{N}})^{b_j} \psi(x)^a$$

$$= \prod_{j=1}^{n} \sum_{x\in\mathbb{F}_q^\times} -t((a^{-1}x)^{\frac{1-q}{N}})^{b_j} \psi(x)$$

$$= \prod_{j=1}^{n} t(a^{\frac{q-1}{N}})^{b_j} \prod_{j=1}^{n} \sum_{x\in\mathbb{F}_q^\times} -t(x^{\frac{1-q}{N}})^{b_j} \psi(x)$$

$$= \prod_{j=1}^{n} g(v, b_j) \qquad (3.2)$$

since $\sum_{j=1}^{n} b_j = 0 \bmod N$.

This suffices when $n > 2$. When $n = 2$, we have to show $\prod_{j=1}^{n} g(v, b_j) \in \mathbb{Q}_l(\zeta_N)^+$. For this, it suffices to take a $\sigma \in G_{\mathbb{Q}_l(\zeta_N)^+}$ such that $\sigma(\zeta_N) = \zeta_N^{-1}$

and $\sigma(\zeta_p) = \zeta_p$, and

$$\sigma(\prod_{j=1}^{2} g(v, b_j)) = \sigma(\prod_{j=1}^{2} \sum_{x \in \mathbb{F}_q^\times} -t(x^{\frac{1-q}{N}})^{b_j} \psi(x))$$

$$= \prod_{j=1}^{2} \sum_{x \in \mathbb{F}_q^\times} -t(x^{\frac{1-q}{N}})^{-b_j} \psi(x)$$

$$= \prod_{j=1}^{2} \sum_{x \in \mathbb{F}_q^\times} -t(x^{\frac{1-q}{N}})^{b_j} \psi(x)$$

$$= \prod_{j=1}^{2} g(v, b_j) \qquad (3.3)$$

because $\{b_1, b_2\} = \{-b_1, -b_2\}$.

Therefore, we deduce that $\det V[\lambda](G_F)$ lands in $(\mathbb{F}_{l^2}^\times k(\lambda)^\times)^n$.

$\square$

We now use the comparison theorems to deduce some $p$-adic Hodge theoretic properties of some $V_{\lambda, t}$. Before doing that, let us fix some notation, following [4].

Let $\mathcal{H}_{dR}$ denote the degree $N - 2$ relative de Rham cohomology of $Y$:

$$\mathcal{H}_{dR} = \mathcal{H}_{dR}^{N-2}(Y/(T_0 \times \mathbb{Q}(\zeta_N))).$$

It is a locally free sheaf over $T_0 \times \mathbb{Q}(\zeta_N)$ with a decreasing filtration $F^j \mathcal{H}_{dR}$ by local direct summands. For $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, let $\mathcal{H}_{dR,\sigma}$, $F^j \mathcal{H}_{dR,\sigma}$ be the "twist" of $\mathcal{H}_{dR}$, $F^j \mathcal{H}_{dR}$ respectively:

$$\mathcal{H}_{dR,\sigma} = \mathcal{H}_{dR} \otimes_{\sigma^{-1}, \mathbb{Q}(\zeta_N)} \mathbb{Q}(\zeta_N), \ F^j \mathcal{H}_{dR,\sigma} = F^j \mathcal{H}_{dR} \otimes_{\sigma^{-1}, \mathbb{Q}(\zeta_N)} \mathbb{Q}(\zeta_N)$$

$H_0$ acts on $\mathcal{H}_{dR}$, $F^j \mathcal{H}_{dR}$ in the usual way. Let $V_{dR,\sigma}$, $F^j V_{dR,\sigma}$ denote the $\chi$ eigenspace of $\mathcal{H}_{dR,\sigma}$, $F^j \mathcal{H}_{dR,\sigma}$:

$$V_{dR,\sigma} = (\mathcal{H}_{dR,\sigma})^{\chi, H_0}, \ F^j V_{dR,\sigma} = (F^j \mathcal{H}_{dR,\sigma})^{\chi, H_0}$$

where we view $\mathcal{H}_{dR,\sigma}$, $F^j \mathcal{H}_{dR,\sigma}$ as $\mathbb{Q}(\zeta_N)$ vector space by acting on the right. Let $\mathrm{gr}^j V_{dR,\sigma} = F^j V_{dR,\sigma} / F^{j+1} V_{dR,\sigma}$ be the associated graded pieces.

Again $H$ acts on $\mathcal{H}_{\mathrm{dR},\sigma,0}$, $F^j \mathcal{H}_{\mathrm{dR},\sigma,0}$, and we have:

$$V_{\mathrm{dR},\sigma,0} = \bigoplus_{i=1}^{N} V_{\mathrm{dR},\sigma,i}, \quad F^j V_{\mathrm{dR},\sigma,0} = \bigoplus_{i=1}^{N} F^j V_{\mathrm{dR},\sigma,i}$$

here $V_{\mathrm{dR},\sigma,i}$, $F^j V_{\mathrm{dR},\sigma,i}$ are the subspace of $V_{\mathrm{dR},\sigma,0}$, $F^j V_{\mathrm{dR},\sigma,0}$ resp. where $H$ acts by:

$$\xi \to \prod_{j=1}^{N} \xi_j^{a_j+i}.$$

and we let $\mathrm{gr}^j V_{\mathrm{dR},\sigma,i} = F^j V_{\mathrm{dR},\sigma,i}/F^{j+1} V_{\mathrm{dR},\sigma,i}$ be the associated graded pieces. Let $\lambda$ be a prime of $\mathbb{Z}[\zeta_N]$ of characteristic $l$ and $v$ be a place of $F$ over $l$. Here $\lambda$ is the prime of the coefficients field as before and $v$ is the place we will restrict to in the $p$-adic Hodge theory setting. We further let $w$ be the place of $\mathbb{Z}[\zeta_N]$ below $v$. Now if $t \in T_0(F_v)$, for an embedding $\sigma : F_v \hookrightarrow \overline{\mathbb{Q}(\zeta_N)}_\lambda$, by the etale comparison theorem, we have

$$((H_{et}^{N-2}(Y_t \times \overline{F_v}, \mathbb{Z}[\zeta_N]_\lambda) \otimes_{\mathbb{Z}[\zeta_N]_\lambda} \overline{\mathbb{Q}(\zeta_N)}_\lambda) \otimes_{\sigma,F_v} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{F_v}/F_v)}$$
$$\cong H_{\mathrm{dR}}^{N-2}(Y_t/F_v) \otimes_{F_v,\sigma} \overline{\mathbb{Q}(\zeta_N)}_\lambda$$

as filtered vector space. Taking the $\chi$ eigenspace of $H_0$ action on both sides gives (notice the twist):

$$((V_{\lambda,t} \otimes_{\mathbb{Z}[\zeta_N]_\lambda} \overline{\mathbb{Q}(\zeta_N)}_\lambda) \otimes_{\sigma,F_v} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{F_v}/F_v)} \cong V_{\mathrm{dR},\sigma_{|\mathbb{Q}(\zeta_N)},t} \otimes_{F_v,\sigma} \overline{\mathbb{Q}(\zeta_N)}_\lambda$$

as filtered vector space.

Similarly, for $i \in \{1, 2, \ldots, N\}$ and $\sigma : \mathbb{Q}(\zeta_N)_w \hookrightarrow \overline{\mathbb{Q}(\zeta_N)}_\lambda$, view $0 \in T_0(\mathbb{Q}(\zeta_N)_w)$, we have:

$$((V_{\lambda,i} \otimes_{\mathbb{Z}[\zeta_N]_\lambda} \overline{\mathbb{Q}(\zeta_N)}_\lambda) \otimes_{\sigma,\mathbb{Q}(\zeta_N)_w} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{\mathbb{Q}(\zeta_N)_w}/\mathbb{Q}(\zeta_N)_w)}$$
$$\cong V_{\mathrm{dR},\sigma_{|\mathbb{Q}(\zeta_N)},i} \otimes_{\mathbb{Q}(\zeta_N)_w,\sigma} \overline{\mathbb{Q}(\zeta_N)}_\lambda$$

For $a \in \mathbb{Z}/N\mathbb{Z}$, we will write $\bar{a}$ as the representative element in the range $\{1, 2, \ldots, N\}$ of $a$. Let $\tau_0 : \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$ be the embedding : $\zeta_N \mapsto e^{2\pi i/N}$. Assume $\sigma^{-1}(\zeta_N) = \zeta_N^a$.

**Lemma 3.10** *Under the notation and assumption above, we have*

(1) *$V_{dR,\sigma,i} \neq (0)$ only when $i \in \{b_1, \ldots, b_n\}$. And for each such $i$, $V_{dR,\sigma,i}$ is a one-dimensional $\mathbb{Q}(\zeta_N)$- vector space and $\mathrm{gr}^j V_{dR,\sigma,i} \neq 0$ only when*

$$j = M(a) + \#\{b \in \{b_1, \ldots, b_n\} : \overline{ab} < \overline{ai}\}$$

*here $M(a)$ is some constant determined by a.*

(2) *$\mathrm{gr}^j V_{dR,\sigma}$ is locally free of rank 1 over $T_0 \times \mathbb{Q}(\zeta_N)$ when $M(a) \leq j \leq M(a) + n - 1$ and is (0) otherwise. $V_{dR,\sigma}$ is a locally free sheaf over $T_0 \times \mathbb{Q}(\zeta_N)$ of rank n.*

*Proof* Base change to $\mathbb{C}$ gives that

$$\mathrm{gr}^j V_{dR,\sigma,i} \otimes_{\mathbb{Q}(\zeta_N), \tau_0\sigma^{-1}} \mathbb{C} \cong H^{j,N-2-j}(Y_0(\mathbb{C}), \mathbb{C})_{(a(a_1+i),\ldots,a(a_N+i))}$$

where we define $Y(\mathbb{C})$ via the embedding $\tau_0$ and right hand side of the isomorphism is defined as the eigenspace of $H^{j,N-2-j}(Y_0(\mathbb{C}), \mathbb{C})$ where $H$ acts by $\xi \to \prod_{j=1}^N \xi_j^{a(a_j+i)}$. Proposition 7.4 and 7.6 of [7] gives that right hand side is nonzero if and only if

- indices $a(a_1 + i), \ldots, a(a_N + i)$ are all nonzero mod $N$
- and

$$j + 1 = (a(a_1 + i) + \ldots + a(a_N + i))/N,$$

i.e. $i \in \{b_1, \ldots, b_n\}$ and we derive the formula of $j$ for a fixed such $i$ as below. For $1 \leq d < N$, let

$$j(d) = (\overline{aa_1 + d} + \ldots + \overline{aa_N + d})/N - 1.$$

Note that the nonzero $a_i$ only appears once in the sum. Then $j(d + 1) = j(d) + 1$ if $d \equiv ab_j \bmod N$ for some $b_j$ (in this case, none of $\overline{aa_i + d}$ is $N$), and $j(d + 1) = j(d)$ if otherwise (in this case, exactly one of $\overline{aa_i + d}$ is $N$). Use this formula to induct, we see that taking $M(a) = j(1)$ gives the formula in (1).

Since $\mathrm{gr}^j V_{dR,\sigma}$ is locally free, it suffice to look at the fibre over 0. Lining up $\overline{ab_i}$ in increasing order, we see that the $j$ such that $\mathrm{gr}^j V_{dR,\sigma,0} \neq 0$ are precisely $M(a), \ldots, M(a) + n - 1$. (2) follows immediately. $\square$

**Lemma 3.11** *Under the notation and assumption above, and let $t \in F_v$ as a point in $T_0(F_v)$, we have*

(1) $V_{\lambda,t}$ is a de Rham representation of $G_F$. For $\sigma : F_v \hookrightarrow \overline{\mathbb{Q}(\zeta_N)}_\lambda$, if $\sigma^{-1}(\zeta_N) = \zeta_N^a$, then the Hodge-Tate weight of $V_{\lambda,t} \otimes_{\mathbb{Z}[\zeta_N]_\lambda} \overline{\mathbb{Q}(\zeta_N)_\lambda}$ with respect to $\sigma$ are

$$\{M(a), \ M(a)+1, \ldots, M(a)+n-1\}.$$

(2) If $t \in \mathcal{O}_{F_v}$ and $t^N - 1 \notin \mathfrak{m}_{F_v}$, then $V_{\lambda,t}$ is crystalline.

(3) If $l \equiv 1 \bmod N$ and $t \in \mathfrak{m}_{F_v}$, then $V_{\lambda,t}$ is ordinary of weight $(\lambda_{\tau,i})$ with $\lambda_{\tau,i} = M(a_\tau)$, $\forall i$, where $a_\tau$ satisfy $\tau^{-1}(\zeta_N) = \zeta_N^{a_\tau}$.

*Proof* (1) is clear from the comparison theorem and Lemma 3.10.

(2) follows from the fact that these $Y_t$ have good reduction modulo the maximal ideal of $F$.

(3) we observe that since $t \in \mathfrak{m}_F$,

$$(V_{\lambda,0} \otimes_{\tau,F} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{F}/F)} \cong (V_{\lambda,t} \otimes_{\tau,F} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{F}/F)}$$

as $\phi$-module because they can both be written as the $\chi$-eigenspace of the crystalline cohomology of the reduction of $Y_0$. Moreover, the Hodge-Tate weights of $V_{\lambda,0}$ and $V_{\lambda,t}$ are the same by (1). Thus by Lemma 2.4, $V_{\lambda,t}$ is ordinary of weight $(M(a_\tau))$ if and only if $V_{\lambda,0}$ is ordinary of weight $(M(a_\tau))$ .

Recall $V_{\lambda,0} = \oplus_{i=1}^N V_{\lambda,i}$ as $G_{\mathbb{Q}(\zeta_N)_w} = G_{\mathbb{Q}_l}$ representation. They are both $\mathbb{Q}(\zeta_N)_\lambda = \mathbb{Q}_l$ vector space. Since

$$((V_{\lambda,i} \otimes_{\mathbb{Z}[\zeta_N]_\lambda} \overline{\mathbb{Q}(\zeta_N)}_\lambda) \otimes_{\tau,\mathbb{Q}(\zeta_N)_w} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{\mathbb{Q}(\zeta_N)_w}/\mathbb{Q}(\zeta_N)_w)}$$
$$\cong V_{\mathrm{dR},\tau_{|\mathbb{Q}(\zeta_N)},i} \otimes_{\mathbb{Q}(\zeta_N)_w,\tau} \overline{\mathbb{Q}(\zeta_N)}_\lambda$$

$V_{\lambda,i}$ is 1-dimensional when $i \in \{b_1, \ldots, b_n\}$ and has Hodge-Tate weight $M(a_\tau) + \#\{b \in \{b_1, \ldots, b_n\} : \overline{a_\tau b} < \overline{a_\tau i}\}$ in this case. Thus

$$V_{\lambda,0} \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{i=0}^{n-1} \overline{\mathbb{Q}}_l(-M(a_\tau) - i)$$

as $I_{\mathbb{Q}(\zeta_N)_w} = I_{\mathbb{Q}_l}$-representation. Therefore (3) follows. $\square$

**Lemma 3.12** *Under the notation and assumption above, and let $\lambda'$ be a place of $\mathbb{Q}(\zeta_N)$ above $l'$, where $l' \neq l$ is an odd prime not dividing $N$. Then view $V_{\lambda',t}$ as a $G_{F_v}$ representation, consider the Weil-Deligne representation $WD(V_{\lambda',t}) = (r_1, N)$ associated to it. We have that the operator $N$ is maximally nilpotent.*

*Proof* The same argument as in part 2 of Lemma 1.15 of [10] gives that $I_{F_v}$ acts on $V_{\lambda',t}$ as $\exp(t_{F_v} N)$, where $t_{F_v}$ is the projection onto the $\mathbb{Z}_{l'}$-factor in

the tame inertia group of $F_v$ and $N$ is a maximally nilpotent operator because the monodromy around $\infty$ of the local system $V_{B,t}$ is maximally unipotent. Now the claim follows from the definition of the Weil-Deligne representation associated to a Galois representation. □

*Remark 3.13* We remark that the proof of the main theorem of [14] shows that when $v(t) < 0$, $V_{\lambda,t}$ (as a $G_{F_v}$ representation) is semistable whose associated Weil-Deligne representation also has a maximally nilpotent operator $N$. This should come as no surprise because of the famous conjecture that geometric families of Galois representations are strongly compatible and the above lemma. However, this conjecture being largely unknown, we cannot deduce one from the other between the two maximally nilpotence of the operators $N$ mentioned above. The full strength of the main result of [14] is not needed in this paper, but we include this remark here because we feel it could be of separate interest.

## 4 Proof of main results theorem 1.1 and theorem 1.4

We fix a non-CM elliptic curve $E/\mathbb{Q}$. For any prime $l'$, let $\overline{r}_{E,l'}$ be the $G_{\mathbb{Q}}$ representation $H^1_{et}(E, \mathbb{F}_{l'})$. Write $n = l^a m, l \nmid m$

We could find (by Lemma 2.3) a positive integer $N$ satisfying the following properties related only to $\overline{r}$, $F^{av}$ and $n$ as given in Theorem 1.1.

- $N$ is odd, and is not divisible by any prime factors of $ln$, any prime that is ramified in $F^{av}$ and $F^{\ker \overline{r}}$ and any prime where the elliptic curve $E$ has bad reduction.
- $N > 100n + 100$
- $\mathbb{F}_{l^2}\mathbb{F}' \subset \mathbb{F}_l(\zeta_N)$ , where $\mathbb{F}'$ is the finite field generated by all the $m$-th roots(hence n-th roots) of elements in the field $\mathbb{F}_{l^s}$ we choose such that the residual representation $\overline{r} : G_F \to GL_n(\mathbb{F}_{l^s})$. And when $n = 2$, we further want that $\mathbb{F}_l(\zeta_N) = \mathbb{F}_{l^r}$ for some $r$ even and $\mathbb{F}_{l^2}\mathbb{F}' \subset \mathbb{F}_l(\zeta_N)^+$. These all amounts to the condition that the smallest positive integer $r$ such that $N \mid l^r - 1$ is divisible by certain integers.
- Let $\mathbb{F}_l(\zeta_N) = \mathbb{F}_{l^r}$. When $r$ is even, we have $N \nmid l^{r/2} + 1$.

Set $F^{avoid}$ to be the normal closure over $\mathbb{Q}$ of $F^{av}\overline{F}^{\ker \overline{r}}(\zeta_l)$. Thus by the condition above, $\mathbb{Q}(\zeta_N)$ and $F^{avoid}$ are linearly disjoint over $\mathbb{Q}$, since any rational prime $p$ that is ramified in their intersection has to divide $N$ while also ramified to $F^{avoid}$. Such prime does not exist, so their intersection is unramified over $\mathbb{Q}$ and thus must be $\mathbb{Q}$. Hence $F^{avoid}$ and $F(\zeta_N)$ are linearly disjoint over $F$. Following the proof of Corollary 7.2.4 of [1], we can prove the following statement:

**Proposition 4.1** *In the above notation, there exists a rational prime $l'$ such that:*

- $l' \equiv 1 \bmod N$.
- $l' > 2ln + 5$ *and is unramified in $F$.*
- $\bar{r}_{E,l'}(G_{\widetilde{F}}) = GL_2(\mathbb{F}_{l'})$, *here $\widetilde{F}$ is the normal closure of $F$ over $\mathbb{Q}$.*
- $\exists \sigma \in G_F - G_{F(\zeta_{l'})}$ *such that $\bar{r}_{E,l'}(\sigma)$ is a scalar.*
- $E$ *has good ordinary reduction at $l'$.*

*And there exists a finite Galois extension $F_2^{avoid}/\mathbb{Q}$ and a finite totally real Galois extension $F^{suff}/\mathbb{Q}$ unramified above the prime divisors of $N$ such that:*

- $F_2^{avoid} \cap F^{avoid} = \mathbb{Q}$,
- $F^{suff} \cap F^{avoid} F_2^{avoid} = \mathbb{Q}$,
- $\overline{\mathbb{Q}}^{\ker \bar{r}_{E,l'}} \subset F_2^{avoid}$,
- $F^{avoid}$ *and $F_2^{avoid}$ are unramified above prime divisors of $N$.*

*and for any finite totally real extension $F'/F^{suff}$ such that $F' \cap F_2^{avoid} = \mathbb{Q}$, $Symm^{n-1}r_{E,l'}|_{G_{F'}}$ is automorphic .*

*Proof* We first pick an $l'$ satisfying the listed properties. This can be done because the first condition give a set of primes of positive density. The second, third and fifth condition exclude a set of primes of density 0(the third by [16] and the fifth by [15] Theorem 20), while the fourth condition follows from the second and the third.(Just pick a $u \in \mathbb{F}_{l'}^{\times}$ with $u^2 \neq 1$ and $\sigma \in G_{\widetilde{F}}$ such that $\bar{r}_{E,l'}(\sigma) = u$.)

Carry out the proof of Corollary 7.2.4 of [1] to $F^{avoid} = F^{avoid}, E = E, \mathcal{M} = \{n-1\}, \mathcal{L} = \{$the prime divisors of $N\}$ and take the $l$ in the proof to be the rational prime $l'$ we just picked. Note that the properties of $l'$ listed in our proposition implies all the properties of $l$ needed in the first paragraph of proof of Corollary 7.2.4 of [1].

Inspecting the proof closely would give that the additional properties (the third and fourth of the lower bullet list) also hold:

The third property follows from the choice of $F_2^{avoid} = F_1^{avoid} L_3$ in the 4th line of page 208 and $L_3 = L_2 \overline{\mathbb{Q}}^{\ker \bar{r}_{E,l'}}$ in the second from last paragraph of page 206 of [1].

For the fourth property, $F^{avoid}$ unramified over $\mathcal{L}$ follows from the choice of $N$ and $F_2^{avoid}$ unramified above $\mathcal{L}$ follows from

1. $L_3$ unramified above $\mathcal{L}$ since $\overline{\mathbb{Q}}^{\ker \bar{r}_{E,l'}}$ is unramified above $\mathcal{L}$ and that each $\overline{\mathbb{Q}}^{\ker \operatorname{Ind}_{G_\mathbb{Q}}^{G_L} \bar{\psi}_m}$ is unramified over $\mathcal{L}$ ($\psi_m$ is unramified over $\mathcal{L}$, see the properties of $\psi_m$ in the beginning of Page 182 and $L$ is also unramified above $\mathcal{L}$, see the paragraph before the last paragraph in Page 181) gives that their composite $L_2$ (page 182) is unramified over $\mathcal{L}$.

2. $F_1^{\mathrm{avoid}}$ is obtained from applying Proposition 7.2.3 of [1] to $F = F_0 = \mathbb{Q}$ (the $F$ is that in the proposition, not our $F$), $\{r_m, \ m \in \mathcal{M}\}$, $\mathcal{L}$ and $F^{\mathrm{avoid}} L_3$ (See the second paragraph of Page 183). In terms of the proof of Proposition 7.2.3 of [1], $F_1^{\mathrm{avoid}} = \overline{\mathbb{Q}}^{\ker \prod_i \overline{r}_i'}(\zeta_{l'})$ with $\overline{r}_i'$ unramified above $\mathcal{L}$ and $l' \notin \mathcal{L}$ (See the last line of Page 180). $\qquad\square$

Note that the condition on $N, l, l'$ guarantees the hypothesis of $N, l$ in Sect. 3 are all satisfied, and $l'$ satisfy the same hypothesis as $l$ with respect to the same $N$. In particular, the conditions of Proposition 3.8 are satisfied.

Now, apply Lemma 2.1 to $F = F$, $M = F(\zeta_N)$, $F_0 = F^{\mathrm{avoid}} F_2^{\mathrm{avoid}} F^{\mathrm{suff}}(\zeta_N)$, to see we may take a finite CM Galois extension $E/F$ with $E = LM$ for some totally real Galois extension $L/\mathbb{Q}$ such that $L$ and $F_0$ are linearly disjoint over $\mathbb{Q}$, and that we may find some characters $\chi_1 : \mathrm{Gal}(\overline{F}/E) \to (\mathbb{Z}[\zeta_N]/\lambda)^\times$ and $\chi_2 : \mathrm{Gal}(\overline{F}/E) \to (\mathbb{Z}[\zeta_N]/\lambda')^\times$ such that $(\overline{\chi_1} \times \overline{\chi_2})^n \cong (\det V[\lambda\lambda']) \otimes \det(\overline{r} \times \mathrm{Symm}^{n-1} r_{E,l'})^\vee$ as $G_E$-module. The condition of Lemma 2.1 is verified below.

On the side of characteristic $l$, fix a prime $\lambda$ of $\mathbb{Q}(\zeta_N)$ ($\mathbb{Q}(\zeta_N)^+$ when $n = 2$) and denote the residue field by $k(\lambda)$. Note that the condition of $N$ in the beginning of this section gives that $\det \overline{r}$ actually has image landing in $(k(\lambda)^\times)^n$. We also assumed that $\mathbb{F}_{l^2} \subset k(\lambda)$, $n \mid \#(k(\lambda)^\times)$. Hence, applying Lemma 3.9 to the field $F(\zeta_N)$, we see that on the characteristic $l$ side, we have $\det V[\lambda] \otimes (\det \overline{r})^\vee$ (as a $G_{F(\zeta_N)}$ representation) has image in $(k(\lambda)^\times)^n$.

On the other side of characteristic $l'$, fixing a prime $\lambda'$ of $\mathbb{Z}[\zeta_N]$ ($\mathbb{Z}[\zeta_N]^+$ when $n = 2$) over $l'$, we have that $\det \mathrm{Symm}^{n-1} \overline{r}_{E,l'} = (\chi_{\mathrm{cyc}})^{n(n-1)/2}$, which have image in $\mathbb{F}_{l^2}^\times$. Applying Lemma 3.9 to the prime $l'$ to see that $(\det \mathrm{Symm}^{n-1} \overline{r}_{E,l'})^\vee \otimes (\det V[\lambda'])$ (as a $G_{F(\zeta_N)}$ representation) has image in $(k(\lambda')^\times)^n$.

Let $W$ be the $\mathbb{Z}[\zeta_N]/\lambda\lambda'$-module with a $G_E$ action given by the representation $(\overline{\chi_1} \otimes \overline{r}) \times (\overline{\chi_2} \otimes \mathrm{Symm}^{n-1}\overline{r}_{E,l'})$. The isomorphism $(\overline{\chi_1} \times \overline{\chi_2})^n \cong (\det V[\lambda\lambda']) \otimes \det(\overline{r} \times \mathrm{Symm}^{n-1} r_{E,l'})^\vee$ induces an isomorphism

$$\phi : \bigwedge^n W_{(T_0)_E} \to \bigwedge^n V[\lambda\lambda']$$

. In this way, the moduli functor $T_W$ is well-defined by $\phi$ over $E$.

We see that the conditions of Proposition 3.8 are satisfied for $N$ and $l, l'$. Thus $T_W$ is geometrically connected.

Note that $F^{\mathrm{avoid}} F_2^{\mathrm{avoid}} F^{\mathrm{suff}}$ and $M = F(\zeta_N)$ are linearly disjoint over $F$ because $F^{\mathrm{avoid}} F_2^{\mathrm{avoid}} F^{\mathrm{suff}}$ and $\mathbb{Q}(\zeta_N)$ linearly disjoint over $\mathbb{Q}$, which in turn comes from $F^{\mathrm{avoid}}$, $F_2^{\mathrm{avoid}}$, $F^{\mathrm{suff}}$ all unramified over the prime divisors of $N$.

Since $L$ is linearly disjoint with $F_0$ over $\mathbb{Q}$ and $M \subset F_0$, we have that $E = LM$ and $F_0$ are linearly disjoint over $M$. Now $E$ and $F^{\mathrm{avoid}} F_2^{\mathrm{avoid}} F^{\mathrm{suff}}$

are linearly disjoint over $F$, because $E \cap F^{\text{avoid}} F_2^{\text{avoid}} F^{\text{suff}} = LM \cap F_0 \cap F^{\text{avoid}} F_2^{\text{avoid}} F^{\text{suff}} = M \cap F^{\text{avoid}} F_2^{\text{avoid}} F^{\text{suff}} = F$

We will need a theorem of Moret-Bailly from [10].

**Proposition 4.2** *Let $F$ be a number field and let $S = S_1 \sqcup S_2 \sqcup S_3$ be a finite set of places of $F$, so that every element of $S_2$ is non-archimedean. Suppose that $T/F$ is a smooth, geometrically connected variety. Suppose also that*

- *for $v \in S_1$, $\Omega_v \subset T(F_v)$ is a non-empty open subset (for the $v$- topology)*
- *for $v \in S_2$, $\Omega_v \subset T(F_v^{nr})$ is a non-empty open $\text{Gal}(F_v^{nr}/F_v)$- invariant subset.*
- *for $v \in S_3$, $\Omega_v \subset T(\overline{F}_v)$ is a non-empty open $\text{Gal}(\overline{F}_v/F_v)$- invariant subset.*

*Suppose finally that $H/F$ is a finite Galois extension. Then there is a finite Galois extension $F'/F$ and a point $P \in T(F')$ such that:*

- *$F'/F$ is linearly disjoint from $H/F$*
- *every place $v$ of $S_1$ splits completely in $F'$ and if $w$ is a prime of $F'$ above $v$, then $P \in \Omega_v \subset T(F'_w)$*
- *every place $v$ of $S_2$ is unramified in $F'$ and if $w$ is a prime of $F'$ above $v$, then $P \in \Omega_v \cap T(F'_w)$*
- *if $w$ is a prime of $F'$ above some $v \in S_3$, then $P \in \Omega_v \cap T(F'_w)$.*

Let $F^+ \subset F$, $E^+ \subset E$ be the maximal totally real subfield respectively. We apply Proposition 4.2 to the smooth geometrically connected variety $T = \text{Res}_{\mathbb{Q}}^{EF^{\text{suff}}} T_W$ defined over $\mathbb{Q}$. We take $H = F_0 L = F_0 E$.

We take $S_1 = \{\infty\}$, $S_2 = \emptyset$ and $S_3 = \{l, l'\}$. For $v \in S_1$, we take $\Omega_v = \text{Res}_{\mathbb{Q}}^{EF^{\text{suff}}} T_W(\mathbb{R})$, i.e. the whole set which is clearly open and non-empty since each copy of $T_W(\mathbb{C})$ are non-empty. For $v \in S_3$, there exists an algebraic morphism $p : T \to \text{Res}_{\mathbb{Q}}^{EF^{\text{suff}}} T_0$ and we define

$$
\Omega_{l,0} = \{t = (t_\tau) \in \text{Res}_{\mathbb{Q}}^{EF^{\text{suff}}} T_0(\overline{\mathbb{Q}}_l)
$$
$$
= \prod_{\tau: EF^{\text{suff}} \hookrightarrow \overline{\mathbb{Q}}_l} T_{0,\tau}(\overline{\mathbb{Q}}_l) \mid v_l(t_\tau) < 0, \forall \tau\},
$$
$$
\Omega_{l',0} = \{t = (t_\tau) \in \text{Res}_{\mathbb{Q}}^{EF^{\text{suff}}} T_0(\overline{\mathbb{Q}}_{l'})
$$
$$
= \prod_{\tau: EF^{\text{suff}} \hookrightarrow \overline{\mathbb{Q}}_{l'}} T_{0,\tau}(\overline{\mathbb{Q}}_{l'}) \mid v_{l'}(t_\tau) > 0, \forall \tau\}
$$

and we define $\Omega_l = p^{-1}(\Omega_{l,0})$, $\Omega_{l'} = p^{-1}(\Omega_{l',0})$. Both sets are clearly open, non-empty and Galois invariant.

Hence, we get a finite totally real Galois extension $L'/\mathbb{Q}$ with $L'$ linearly disjoint with $F_0 L$ over $\mathbb{Q}$ and a point $t \in T_0(L'EF^{\text{suff}})$ (because $L'$ and $EF^{\text{suff}}$ are linearly disjoint over $\mathbb{Q}$) such that if we denote $L'EF^{\text{suff}}$ by $F'$ and $L'E^+ F^{\text{suff}}$ by $(F')^+$ then

- $F' \supset EF^{\text{suff}}$ is a CM Galois extension over $F$,
- $\overline{\chi_1}^{-1} V[\lambda]_t \cong \overline{r}|_{G_{F'}}$,
- $\overline{\chi_2}^{-1} V[\lambda']_t \cong \overline{\text{Symm}^{n-1} r_{E,l'}}|_{G_{F'}}$,
- $v(t) < 0$ for all primes $v|l$ of $F'$,
- $v(t) > 0$ for all primes $v|l'$ of $F'$.

Now $F' \cap F_2^{\text{avoid}} = L'EF^{\text{suff}} \cap F_0 L \cap F_2^{\text{avoid}} = EF^{\text{suff}} \cap F_2^{\text{avoid}}$ because $L'$ and $F_0 L$ are linearly disjoint over $\mathbb{Q}$. Moreover, $EF^{\text{suff}} \cap F_2^{\text{avoid}} = LMF^{\text{suff}} \cap F_0 \cap F_2^{\text{avoid}} = MF^{\text{suff}} \cap F_2^{\text{avoid}}$ because $L$ and $F_0$ are linearly disjoint over $\mathbb{Q}$. Furthermore, $MF^{\text{suff}} \cap F_2^{\text{avoid}} = F^{\text{suff}} F(\zeta_N) \cap F^{\text{suff}} F^{\text{avoid}} F_2^{\text{avoid}} \cap F_2^{\text{avoid}} = F^{\text{suff}} F \cap F_2^{\text{avoid}}$ because $F^{\text{avoid}} F_2^{\text{avoid}} F^{\text{suff}}$ and $\mathbb{Q}(\zeta_N)$ linearly disjoint over $\mathbb{Q}$. Finally, $F^{\text{suff}} F \cap F_2^{\text{avoid}} = F^{\text{suff}} F \cap F^{\text{avoid}} F_2^{\text{avoid}} \cap F_2^{\text{avoid}} = F \cap F_2^{\text{avoid}} \subset F^{\text{avoid}} \cap F_2^{\text{avoid}} = \mathbb{Q}$ because $F^{\text{suff}}$ and $F^{\text{avoid}} F_2^{\text{avoid}}$ are linearly disjoint over $\mathbb{Q}$. Therefore, we conclude that $F' \supset F^{\text{suff}}$ and is linearly disjoint with $F_2^{\text{avoid}}$ over $\mathbb{Q}$, so we see by Proposition 4.1 that $\text{Symm}^{n-1} r_{E,l'}|_{G_{(F')^+}}$ and hence $\text{Symm}^{n-1} r_{E,l'}|_{G_{F'}}$ is automorphic. Since $F'$ and $\overline{\mathbb{Q}}^{\ker \overline{r}_{E,l'}} (\subset F_2^{\text{avoid}})$ are linearly disjoint over $\mathbb{Q}$, we again have

- $\overline{r}_{E,l'}(G_{F'}) \supset SL_2(\mathbb{F}_{l'})$.
- $\exists \sigma \in G_{F'} - G_{F'(\zeta_{l'})}$ such that $\overline{r}_{E,l'}(\sigma)$ is a scalar.

Note that by similar reasoning as the previous paragraph, we have that $F' \cap F^{\text{avoid}} F_2^{\text{avoid}} = L'EF^{\text{suff}} \cap F_0 L \cap F^{\text{avoid}} F_2^{\text{avoid}} = EF^{\text{suff}} \cap F^{\text{avoid}} F_2^{\text{avoid}} = LMF^{\text{suff}} \cap F_0 \cap F^{\text{avoid}} F_2^{\text{avoid}} = MF^{\text{suff}} \cap F^{\text{avoid}} F_2^{\text{avoid}} = F^{\text{suff}} F(\zeta_N) \cap F^{\text{suff}} F^{\text{avoid}} F_2^{\text{avoid}} \cap F^{\text{avoid}} F_2^{\text{avoid}} = F^{\text{suff}} F \cap F^{\text{avoid}} F_2^{\text{avoid}} = F$ and thus $F'$ is linearly disjoint with $F^{\text{av}}$ over $F$ as we wanted in the main theorem.

Let $\chi_2 : G_E \to \mathbb{Q}(\zeta_N)^{\times}$ be the Teichmuller lift of $\overline{\chi}_2$. We would like to apply Theorem 6.1.2 of [1] to $p = l'$, $\rho = V_{\lambda',t} \otimes \chi_2^{-1}$ and $r_\iota(\pi) = \text{Symm}^{n-1} r_{E,l'}|_{G_{F'}}$. Clearly $\overline{\rho} \cong \overline{r_\iota(\pi)}$. For the residual representation $\text{Symm}^{n-1} \overline{r}_{E,l'}|_{G_{F'}}$, the two properties of $\overline{r}_{E,l'}$ listed in 4.1 gives that it is absolutely irreducible and condition (4) of Theorem 6.1.2 of [1] is satisfied. Now apply Lemma 2.6 (1) to $F = F$, $F_1 = F'$, $l = l'$, $\overline{r} = \overline{r}_{E,l'}$ and the fact that $F'$ and $\overline{\mathbb{Q}}^{\ker \overline{r}_{E,l'}}$ are linearly disjoint over $\mathbb{Q}$, to see $(\text{Symm}^{n-1} \overline{r}_{E,l'})(G_{F'(\zeta_{l'})}) = (\text{Symm}^{n-1} \overline{r}_{E,l'})(G_{F(\zeta_{l'})})$ is enormous. Apply Lemma 2.6 (2) to the same situation ($H \subset \widetilde{F} \overline{\mathbb{Q}}^{\ker \overline{r}_{E,l'}} \subset \widetilde{F} F_2^{\text{avoid}} \subset F^{\text{avoid}} F_2^{\text{avoid}}$, thus $H' \subset F^{\text{avoid}} F_2^{\text{avoid}}$ since $F^{\text{avoid}}$ and $F_2^{\text{avoid}}$ are both Galois over $\mathbb{Q}$, and then $F'$ linearly disjoint with $F^{\text{avoid}} F_2^{\text{avoid}}$ over $F$ is satisfied) to

see that $\overline{\rho}$ is decomposed generic. Now $V_{\lambda',t}$ is a regular ordinary representation for any place $v_{l'} \mid l'$ of $F'$ by Lemma 3.11 (3) ($v_{l'}(t) > 0$) and thus so is $\rho$. Our choice of $l'$ gives that $\mathrm{Symm}^{n-1}r_{E,l'}|_{G_{F'}}$ is ordinary as a $G_{F'_{v'}}$ representation for any $l'$-adic place $v'$ of $F'$. Hence we may apply Lemma 5.9 of [8] to see that $\mathrm{Symm}^{n-1}r_{E,l'}|_{G_{F'}}$ is $\iota'$-ordinarily automorphic (since $\mathrm{Symm}^{n-1}r_{E,l'}|_{G_{F'}}$ is clearly polarizable). Theorem 6.1.2 of [1] thus gives that $V_{\lambda',t}$ is automorphic as a $G_{F'}$ representation.

Hence $V_{\lambda,t}$ is automorphic as a $G_{F'}$ representation. Now by the following Lemma 4.3 ($v_l(t) < 0$ for any $v_l \mid l$), we see that $V_{\lambda,t} \otimes \chi_1^{-1}$ is $\iota$-ordinarily automorphic and hence $\overline{r}\ |_{G_{F'}}$ is ordinarily automorphic. This finishes the proof of Theorem 1.1.

**Lemma 4.3** *We assume $t \in F'$ satisfy $v(t) < 0$ for any $l$-adic places $v$ of $F'$ and $V_{\lambda,t}$ is automorphic, then $V_{\lambda,t}$ is $\iota$-ordinarily automorphic.*

*Proof* Write $V_{\lambda,t} = r_{l,\iota}(\pi)$, then the only thing we need to show is that $\pi_v$ is $\iota$-ordinary for any $l$-adic places $v$ of $F'$. Lemma 3.12 shows that $WD(V_{\lambda',t}|_{G_{F'_v}}) \cong WD(r_{l',\iota'}(\pi)|_{G_{F'_v}})$ has a maximally nilpotent $N$ and hence is Steinberg. Now the main theorem of [17] shows that

$$WD(r_{l',\iota'}(\pi)|_{G_{F'_v}})^{\mathrm{Frob-ss}} \prec \iota^{-1}\mathrm{rec}_{F'_v}(\pi_v|\det|_v^{\frac{1-n}{2}}).$$

. But $(n,0,0,\ldots)$ is the largest among the order $\prec$, thus $\mathrm{rec}_{F'_v}(\pi_v|\det|_v^{\frac{1-n}{2}})$ has a maximally nilpotent $N$ and hence $\pi_v$ is Steinberg.

Now we apply a variant of Lemma 5.6 of [8] to conclude $\pi_v$ is $\iota$-ordinary from the fact that $\pi_v$ is Steinberg. Namely, we show that the weight 0 condition is not needed. In the notation of Geraghty, we write $\pi_v \cong \mathrm{Sp}_n(\psi_v|\cdot|^{\frac{1-n}{2}})$. By Lemma 3.11, we have that the Hodge-Tate weight of $V_{\lambda,t}$ has the shape $(\lambda_\tau, \lambda_\tau + 1, \ldots, \lambda_\tau + n - 1)_\tau$, we see that the weight of the automorphic representation $\pi$ must be of form $(-\lambda_\tau)_\tau$, here in terms of Geraghty's notation it means that all $\lambda_{\tau,i} = -\lambda_\tau$ for a fixed $\tau$. Because of this property and in view of Lemma 5.2 of [8], it suffices to show that $\mathrm{val}_l(\iota^{-1}(\psi_v(\det(\alpha_{\varpi_v}^{(n)})))) = \mathrm{val}_l(\prod_{\tau:F'_v\hookrightarrow\overline{\mathbb{Q}}_l}\tau(\varpi_v)^{-n\lambda_\tau})$ (for the other $j \in [0,n]$, it suffices to divide this equality by $n$ and multiply by $j$). Let $\phi_\pi$ be the central character of $\pi$, then the left hand side of the above is just $\mathrm{val}_l(\iota^{-1}(\phi_{\pi,v}(\varpi_v)))$. Now we know that $r_{l,\iota}(\phi_\pi) = \det r_{l,\iota}(\pi) \cdot \chi_{\mathrm{cyc}}^{\frac{n(n-1)}{2}}$, here the $r_{l,\iota}$ on the left hand side is the 1-dimensional Langlands sending algebraic characters to $l$-aidc characters. Thus $r_{l,\iota}(\phi_\pi)$ is an $l$-adic character of Hodge-Tate weight $(n\lambda_\tau)_\tau$, and since the image of $r_{l,\iota}(\phi_\pi)$ are all $l$-adic units, we see that $\mathrm{val}_l(\iota^{-1}(\phi_{\pi,v}(\varpi_v))) = \mathrm{val}_l(\prod_{\tau:F'_v\hookrightarrow\overline{\mathbb{Q}}_l}\tau(\varpi_v)^{-n\lambda_\tau})$ and we are done.

$\square$

*Remark 4.4* Here is a follow-up on Remark 3.13. In the process of proof above, we actually proved that $V_{\lambda,t} \cong r_{l,\iota}(\pi)$ is $\iota$-ordinarily automorphic. Hence by the main local-global compatibility result Theorem 5.5.1 or [1], we see that $V_{\lambda,t}$ as a $G_{F'_v}$ representation is ordinary for any $l$-adic places $v$ of $F$. This gives another way of proving the main result of [14]. However, as we have commented, the attempt of deducing $V_{\lambda,t}$ being $\iota$-ordinarily automorphic from the fact that it is automorphic and ordinary as $l'$-adic representation seems invalid because to the best of the author's knowledge, the local-global compatibility result in this direction assumes the Galois representation to be polarizable.

The author wants to thank Jack Thorne for pointing out this issue (together with Remark 3.13) to him.

For the proof of Theorem 1.4, we know from above that $\overline{\chi_1}^{-1}V[\lambda]_t \cong \bar{r}|_{G_{F'}}$ and $V_{\lambda,t} \otimes \chi_1^{-1}$ is $\iota$-ordinarily automorphic. Thus, in order to apply Theorem 6.1.2 of [1], it suffices to verify that the conditions (3) and (4) of that theorem holds for $\bar{r}|_{G_{F'}}$. Since $F'$ is linearly disjoint with $\overline{F}^{\ker \bar{r}} \subset F^{\mathrm{avoid}}$ over $F$, all conditions except decomposed genericity follows from the corresponding conditions of $\bar{r}$. Now $F' = L'EF^{\mathrm{suff}} = FL'LF^{\mathrm{suff}}(\zeta_N)$ is Galois over $F$. And the Galois closure of $\overline{F}^{\ker \bar{r}}(\zeta_l)$ is linearly disjoint with $F'$ over $\mathbb{Q}$ because this Galois closure is contained in $F^{\mathrm{avoid}}$ and that $L'LF^{\mathrm{suff}}(\zeta_N) \cap F^{\mathrm{avoid}} = L'LF^{\mathrm{suff}}(\zeta_N) \cap LF_0 \cap F^{\mathrm{avoid}} = LF^{\mathrm{suff}}(\zeta_N) \cap F^{\mathrm{avoid}} = LF^{\mathrm{suff}}(\zeta_N) \cap F_0 \cap F^{\mathrm{avoid}} = F^{\mathrm{suff}}(\zeta_N) \cap F^{\mathrm{avoid}} = F^{\mathrm{suff}}(\zeta_N) \cap F^{\mathrm{suff}}F^{\mathrm{avoid}}F_2^{\mathrm{avoid}} \cap F^{\mathrm{avoid}} = F^{\mathrm{suff}} \cap F^{\mathrm{avoid}} = \mathbb{Q}$. Now we may apply Lemma 7.1.7 of [1] to see the decomposed genericity. Hence we also finish the proof of Theorem 1.4.

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